# Year 1 — Calculus of Several Variables (Honors)

Based on lectures by T. Arant
Discussions by J. Murri
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MATH 32AH, Fall Quarter 2024, UCLA

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine. A note of acknowledgement to Dexter Chua, Ph.D. Harvard University for the template.

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**Catalog Description**
Lecture, three hours; discussion, one hour. Enforced requisite: course 31A with grade of B or better. Honors course parallel to course 32A. P/NP or letter grading.

**Textbook Reading**
Calculus and Analysis in Euclidean Space, *Jerry Shurman*

**Contact**
This document is a summary of the notes that I have taken during lectures at UCLA; please note that this lecture note will not necessarily coincide with what you might learn. If you find any errors, don't hesitate to reach out to me below:

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# Contents

# 1 Euclidean Spaces

**Definition 1.** Space
A set of elements *plus* a structure to these elements.

The set, $\mathbb{R}$ is the set of *real numbers.* Let $n$ be a positive integer, (i.e. $n$ is an element of the set $\{1, 2, 3, \dots\}$), $n$-dimensional euclidean space is the set $\mathbb{R}^n$.

$$\mathbb{R}^n = \{(x_1, x_2, \dots) : x_i \in \mathbb{R} \text{ for } i = 1, 2, \dots, n\}$$

i.e., the set of *n-tuples* where each coordinate $x_i$ is a real.

**Note:** The notation, ":" reads, 'such that'. Alternatively, | can also be used to express *such that.*

A quick note on **Set Notation**,

1. $a \in A$ (*a is a member of* set $A$)
2. $a \notin A$ (*a is not a member of* set $A$)

**Remark.** $(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$ if and only if (iff) $x_i = y_i$ $\forall$ (for all) $i = 1, 2, \dots, n$.

Notice that the order of coordinates matter in tuples, that is to say, the order in which elements appear in a tuple is significant and can't be ignored. This is **not** the case for sets. Members of a set are not ordered.

$$\{1, 2, 3\} = \{3, 2, 1\}$$

Sets also don't have *multiplicity*, that is, the number of occurrences of particular elements within a set is not significant.

$$\{1, 1, 1, 2, 2, 3\} = \{1, 2, 3\}$$

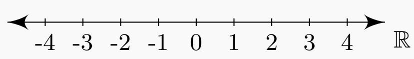
It's possible to conclude that while tuples are ordered and have multiplicity, sets in comparison are **not** ordered and do **not** have multiplicity.

## 1.1 Visualizing Euclidean Spaces

$\mathbb{R}^1$ is technically the set of 1-tuples. $(x_1)$ where $x \in \mathbb{R}$. However, we usually conflate $\mathbb{R}^1$ with $\mathbb{R}$.

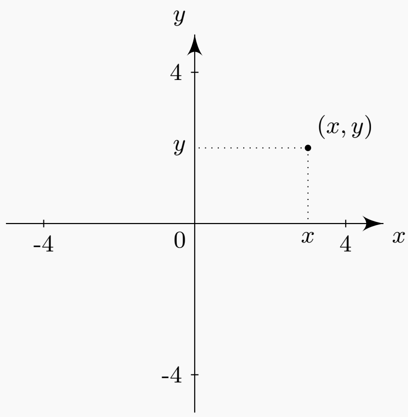
**Visualizing 1-dimensional Euclidean Space**

$\mathbb{R}$ is 1-dimensional euclidean space. Geometrically, it's the real line.



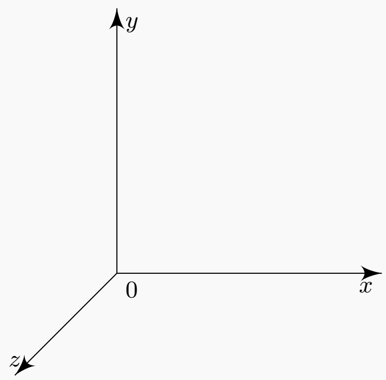
**Visualizing 2-dimensional Euclidean Space**

$\mathbb{R}^2$ is 2-dimensional euclidean space. Geometrically, it's the real plane.




**Visualizing 3-dimensional Euclidean Space**

$\mathbb{R}^3$ is 3-dimensional euclidean space. Geometrically, it's the 3-space.



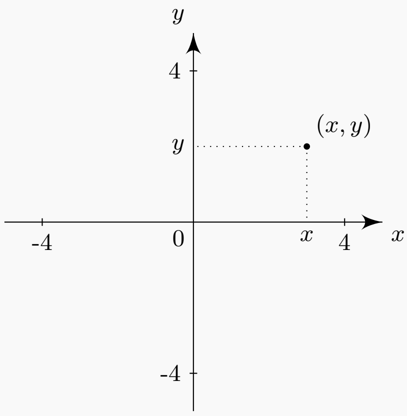

$\mathbb{R}^4$ is also a 4-dimensional euclidean space. However, we can't visually discern $\mathbb{R}^4$ but it is still important. $\mathbb{R}^4$ can be construed as *spacetime*. $(x, y, z, t)$ are the coordinates used to express $\mathbb{R}^4$. In *spacetime*, the coordinates, $(x, y, z)$ express coordinates of position in space, while $t$ expresses the time coordinate. Furthermore, the aforementioned is only a suggested notation, $t$ doesn't necessarily need to be "time" in actuality.

## 1.2 Vectors

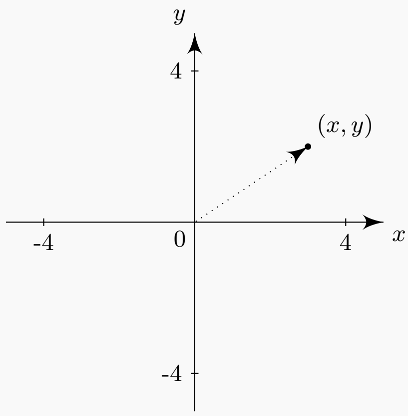
Elements of $\mathbb{R}^n$ are often called (real) vectors. There are two ways to think about vectors,

**Vectors**

**A Point in Space** (E.g. in $\mathbb{R}^2$)




**Arrow with Tail and Head**




**Remark.** Every $\mathbb{R}^n$ has a zero vector or *origin.* $\mathbf{0} = \underbrace{(0, 0, \ldots, 0)}_{n \text{ times}}$

Let's elaborate more on the term, "space". Space doesn't just mean a set, a space, as mentioned before, is a set *and* structure. What is this structure? This structure can be focused on two algebraic operations.

### 1.2.1 Vector Addition

Vector addition, in function notation, is expressed as,

$$\underbrace{+}_{+\text{ is a function}} : \underbrace{\mathbb{R}^n \times \mathbb{R}^n}_{\text{input values, e.g. } x,\ y} \rightarrow \underbrace{\mathbb{R}^n}_{\text{output value, e.g. } x+y}$$

**Remark.** Both of the $\mathbb{R}^n$ in the above function expression are two vectors in $\mathbb{R}^n$.

Vector Addition is defined point-wise (or coordinate-wise).

$$(x_1, \ldots, x_n) \underbrace{+}_{\text{Vector Addition}} (y_1, \ldots, y_n) = \left( x_1 \underbrace{+}_{+\text{ on } \mathbb{R}} y_1, \ldots, x_n + y_n \right)$$

Note that the two '+' symbols in the aforementioned expression are **not** equivalent; they mean different things.

### 1.2.2 Scalar Multplication

The 2nd algebraic operation is *scalar multiplication*, formally written as '$\cdot$' (center dot). In function notation, this is expressed as,

$$\cdot : \underbrace{\mathbb{R}^n}_{\text{scalar value}} \times \underbrace{\mathbb{R}^n}_{\text{vector value}} \rightarrow \mathbb{R}^n$$

If $a$ is a scalar, and $x$ is a vector,

$$a \cdot (x_1, \ldots, x_n) = (ax_1, \ldots, ax_n)$$

**Note:** We don't multiply vectors in $\mathbb{R}^n$ when $n \geq 2$. Expressions such as $xy$ and $\dfrac{y}{x}$ are not possible and should not be written at all. We can only multiply vectors in $\mathbb{R}^1$, not $\mathbb{R}^2$ or above. When we have these two algebraic operations, there are some properties that show how these operations work together.

## 1.3 Vector Space Axioms

The structure, $(\mathbb{R}^n, +, \cdot)$ satisfies the following properties:

(A1) Addition is ***associative***. $\forall\ x, y, z \in \mathbb{R}^n$, $(x + y) + z = x + (y + z)$

(A2) $\mathbf{0}$ (Zero vector) $= \underbrace{(0, \ldots, 0)}_{n \text{ times}}$ is the ***additive identity***. $\forall\ x \in \mathbb{R}^n$, $x + 0 = x$

(A3) Existence of ***additive inverse***. $\forall\ x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$ such that $x + y = 0$ (This $y$ is unique and we write $y = -x$)

(A4) Addition is ***commutative***. $\forall\ x, y \in \mathbb{R}^n$, $x + y = y + x$

(M1) Scalar Multiplication is ***associative***. $\forall\ a, b \in \mathbb{R}$, $x \in \mathbb{R}^n$, $(a \cdot b)\, x = a \cdot (bx)$.

(M2) $\mathbf{1} \in \mathbb{R}$ is the scalar ***multiplicative identity*** $\forall\ x \in \mathbb{R}^n$, $1 \cdot x = x$

(D1) Scalar Multiplication *distributes* over ***scalar addition***. $\forall\ a, b \in \mathbb{R}$, $x \in \mathbb{R}^n$, $(a + b) \cdot x = ax + bx$

(D2) Scalar Multiplication ***distributes*** over *vector addition*. $\forall\ a \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, $a \cdot (x + y) = ax + ay$

**Remark.** If we replace $\mathbb{R}^n$ with an arbitrary set $V \neq \emptyset$ and $(V, +, \cdot)$ satisfy these properties, then $(V, +, \cdot)$ is called a *real vector space*.

**Theorem 1.** *For any $x \in \mathbb{R}^n$, $(-1)\, x = -x$, i.e. $(-1)\, x$ is the additive inverse of $x$.*

*Proof.* We want to check $(-1)\, x + x = \mathbf{0}$

$$\begin{aligned}
(-1)\, x + x &= (-1)\, x + (1)\, x \\
&= (-1 + 1)\, x = 0x \,\text{[Distributing]} \\
&= 0
\end{aligned}$$

□

**Lemma.** For all $x \in \mathbb{R}^n$, $0x = 0$

*Proof.*

$$\begin{aligned}
0x &= (0 + 0)\, x = 0x + 0x \\
0x - 0x &= 0x + 0x - 0x \\
&\implies 0 = 0x
\end{aligned} \tag{1}$$

□

**Remark.** This is just algebra.

Why are (A1) to (A3), (M1), (M2), (D1), (D2) true? That vector addition is commutative easily follows from commutativty of $+$ in $\mathbb{R}$.

**Theorem 2.** *(A4) is true.*

*Proof.* Let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in \mathbb{R}^{\ltimes}$

$$\begin{aligned}
\text{By Definition of Vector Addition}\, (x_1, \ldots, x_n) + (y_1, \ldots, y_n) &= (x_1, \ldots, x_n + y_n) \\
\text{By Commutativty} &= (y_1 + x_1, \ldots, y_n + x_n) \\
\text{By Definition of Vector Addition}\, (y_1, \ldots, y_n) + (x_1, \ldots, x_n)
\end{aligned}$$

∎

## 1.4 Basis of $\mathbb{R}^n$

**Definition 2.** A set of $n$ vectors $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^n$ is a *basis* if for every vector $x \in \mathbb{R}^n$, there exists *unique* scalars $a_1, \ldots, a_n \in \mathbb{R}$ such that $x = a_1 v_1 + \cdots + a_n v_n$.

**Remark.** $a_1 v_1 + \cdots + a_n v_n$ is the linear combination of $v_1, \ldots, v_n$.

By uniqueness, we mean that for any $b_1, \ldots, b_n \in \mathbb{R}$,

$$a_1 v_1 + \ldots a_n v_n = b_1 v_1 + \ldots b_n v_n \implies a_i = b_i$$

for $i = 1, \ldots, n$.

### 1.4.1 The Standard Basis of $\mathbb{R}^n$

**Definition 3.** The standard basis of $\mathbb{R}^n$ is the set $\{e_1, \ldots, e_n\}$ where

$$e_i = \left( 0, \ldots, 0, \overbrace{1}^{i\text{-th component}}, 0, \ldots, 0 \right) \in \mathbb{R}^n$$

In $\mathbb{R}^2$, $e_1 = (1, 0), e_2 = (0, 1)$, in $\mathbb{R}^3, e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$.

**Remark.** Standard basis is not the only basis.

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned}
x = (x_1, \ldots, x_n) &= (x_1, 0, \ldots, 0) + (0, x_2, \ldots, 0) + \cdots + (0, 0, \ldots, x_n) \\
&= x_1 (1, 0, \ldots, 0) + x_2 (0, 1, \ldots, 0) + \cdots + x_n (0, 0, \ldots, 1) \\
&= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n
\end{aligned}$$

**Example 1.** Show $\{(1, 0), (1, 1)\}$ is a basis of $\mathbb{R}^2$.

*Proof.* Let $x \in \mathbb{R}^2$. We want $a, b$ which satisfy

$$\begin{aligned}
&x = (x_1, x_2) = a(1, 0) + b(1, 1) = (a + b, b) \\
&\iff \begin{cases} x_1 = a + b \\ x_2 = b \end{cases} \iff \begin{cases} x_1 - b = a \\ x_2 = b \end{cases} \iff \begin{cases} a = x_1 - x_2 \\ b = x_2 \end{cases}
\end{aligned}$$

Thus, $x = (x_1 - x_2)(1, 0) + x_2 (1, 1)$

For every $x$, we've shown that there is some linear combination determined solely by the vectors of $x$. ∎

**Remark.** Uniqueness also follows from our above work since we showed

$$x = a\,(1,0) + b\,(1,1) \iff a = x_1 - x_2, b = x_2$$

# 2 Geometry: Length & Angle

## 2.1 Length and The Inner Product

**Definition 4.** *The Inner product.*
The inner product (or the dot product) of vectors $x, y \in \mathbb{R}^n$ is $\langle x, y\rangle$ such that

$$\begin{aligned}\langle x, y\rangle &= \langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = \sum_{i=1}^{n} x_i y_i \\ &= x_1y_1 + x_2y_2 + \cdots + x_ny_n\end{aligned}$$

**Example 2.**

$$\begin{aligned}\langle (1,3,-1), (2,0,4)\rangle &= 1\cdot 2 + 3\cdot 0 + (-1)\cdot 4 \\ &= 2 + 0 - 4 = -2\end{aligned}$$

**Remark.** The context of the result from the inner product will be discussed later.

For $x \in \mathbb{R}^n$,

$$\langle x, e_i\rangle = \langle (x_1, \ldots, x_n), \left(0, \ldots, 0, \overbrace{1}^{i\text{-th component}}, 0, \ldots, 0\right)\rangle = x_i$$

**Definition 5.** Vectors $x, y \in \mathbb{R}^n$ are called *orthogonal* if $\langle x, y\rangle = 0$

Note that $\mathbf{0}$ is orthogonal to every vector,

**Example 3.** $(1,1,1,1)$ and $(-1,1,-1,1)$ are orthogonal.

**Theorem 3.** *(Inner Product Properties)*

*(IP1) Positive Definite:* $\forall\ x \in \mathbb{R}^n$, $\langle x, x\rangle \geq 0$
*and only one case* $\langle x, x\rangle = 0 \iff x = 0$

*(IP2) Symmetry:* $\forall\ x, y \in \mathbb{R}^n, \langle x, y\rangle = \langle y, x\rangle$

*(IP3) Bilinearity:* $\forall\ x, x', y, y' \in \mathbb{R}^n, a, b \in \mathbb{R}$

$$\begin{cases}\textit{Linear in 1st component} & \langle x + x', y\rangle = \langle x, y\rangle + \langle x', y\rangle, \langle ax, y\rangle = a\langle x, y\rangle \\ \textit{Linear in 2nd component} & \langle x, y + y'\rangle = \langle x, y\rangle + \langle x, y'\rangle, \langle x, ay\rangle = a\langle x, y\rangle\end{cases}$$

Let's prove (IP1),

*Proof.* $\langle x, x\rangle = x_1^2 + \cdots + x_n^2 \geq 0$. Since $x^2$ is always positve, $\langle x, x\rangle \geq 0$ □

**Remark.** Only case where $\langle x, x\rangle = 0$ is if each $x_i$ for $i = 1, \ldots, n$ is also 0.

**Definition 6.** The modulus (or absolute value or length) of $x \in \mathbb{R}^n$ is

$$|x| = \sqrt{\langle x, x\rangle}$$

In $\mathbb{R}$, $|a| = \sqrt{a^2}$. We are only taking the principle branch, the positive square root. The modulus of $x$ is always real by (IP1).

**Theorem 4.** *(Modulus Properties)*
*(Mod 1)* $\forall\ x \in \mathbb{R}^n,\ |x| \geq 0, |x| = 0 \iff x = 0$
*(Mod 2)* $\forall\ x \in \mathbb{R}^n, a \in \mathbb{R},\ |ax| = |a|\,|x|$

**Theorem 5.** *Cauchy-Bunyakovsky-Schwarz Inequality*

$$\forall x, y \in \mathbb{R}^n, |\langle x, y\rangle| \leq |x| \cdot |y|$$

*Proof.* When $x = 0$, the result is trivial since both sides $= 0$. So we may assume $x \neq 0$. Then, for any $a \in \mathbb{R}$,

$$\begin{aligned}
0 \leq \langle ax - y, ax - y\rangle &= \langle ax, ax - y\rangle + \langle -y, ax - y\rangle \\
&= \langle ax, ax\rangle + \langle ax, -y\rangle + \langle -y, ax\rangle + \langle -y, -y\rangle \\
&= a\langle x, ax\rangle + a\langle x, -y\rangle + \langle -y, ax\rangle + \langle -y, -y\rangle \\
&= a^2\langle x, x\rangle + a\langle x, -y\rangle + a\langle -y, x\rangle + \langle -y, -y\rangle \\
&= a^2\langle x, x\rangle + a\langle x, -y\rangle - a\langle y, x\rangle - \langle y, -y\rangle \\
&= a^2\langle x, x\rangle - a\langle x, y\rangle - a\langle y, x\rangle + \langle y, y\rangle \\
&= a^2\,|x|^2 - 2a\langle x, y\rangle + |y|^2
\end{aligned}$$

It follows that some quadratic polynomial in $a$, $f(a) \geq 0$. Therefore, this polynomial can at most have one real root (or no real roots). This implies that its discriminant is non-positive.

$$\begin{aligned}
D \leq 0 &\implies 4\langle x, y\rangle^2 - 4\,|x|^2\,|y|^2 \leq 0 \\
\langle x, y\rangle^2 - |x|^2\,|y|^2 \leq 0 &\implies \langle x, y\rangle^2 \leq |x|^2\,|y|^2 \\
&\implies |\langle x, y\rangle| \leq |x| \cdot |y|
\end{aligned}$$

□

**Remark.** The Cauchy-Schwarz Inequality is equivalent to,

$$\underbrace{-\,|x| \cdot |y|}_{\text{Lower bound}} \leq \langle x, y\rangle \leq \overbrace{|x| \cdot |y|}^{\text{Upper bound}}$$

The $\leq$ is actually $=$ if and only if one of $x, y$ is a scalar multiple of the other, i.e., $x, y$ are parallel.

**Theorem 6.** *Triangle Inequality*

$$\forall x, y, \in \mathbb{R}^n, |x + y| \leq |x| + |y|$$

*Proof.*

$$\begin{aligned}
|x + y|^2 &= \langle x + y, x + y\rangle \\
\text{(By Linearity in 1st Variable)} &= \langle x, x + y\rangle + \langle y, x + y\rangle > \\
\text{(By Linearity in 2nd Variable)} &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
\text{(By Symmetry)} &= |x|^2 + 2\langle x, y\rangle + |(|\,y)^2 \\
\text{(By Cauchy-Schwarz)} &\leq |x|^2 + 2\,|x| \cdot |y| + |y|^2 = (|x| + |y|)^2 \\
|x + y|^2 &\leq (|x| + |y|)^2
\end{aligned}$$

$$\therefore |x + y| \leq |x| + |y|$$

□

**Remark.** For any $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$ for $1 \leq k \leq n$,

$$|x_1 + (x_2 + \cdots + x_k)|$$

$$\text{Induction} \begin{cases} \leq |x_1| + |x_2 + (\cdots + x_k)| \\ \leq |x_1| + |x_2| + |x_3 + (\cdots + x_k)| \\ \leq \cdots \leq |x_1| + |x_2| + \cdots + |x_k| \end{cases}$$

$\therefore$

$$|x_1 + \ldots + x_k| \leq |x_1| + \ldots + |x_k|$$

**Remark.** There are important variants of the triangle inequality,

$$||x| + |y|| \leq |x - y| \leq |x| + |y|$$

## 2.2 Angles

By Cauchy-Schwarz,

$$-|x| \cdot |y| \leq \langle x, y \rangle \leq |x| \cdot |y|$$

when $x \neq \mathbf{0}$ & $y \neq \mathbf{0}$ (so $|x| \neq 0$, $|y| \neq 0$)

$$\frac{-|x| \cdot |y|}{|x| \cdot |y|} \leq \frac{\langle x, y \rangle}{|x| \cdot |y|} \leq \frac{|x| \cdot |y|}{|x| \cdot |y|} \iff -1 \leq \frac{\langle x, y \rangle}{|x| \cdot |y|} \leq 1$$

There's a unique angle $0 \leq \theta_{x,y} \leq \pi$ such that $\cos \theta_{x,y} = \dfrac{\langle x, y \rangle}{|x| \cdot |y|}$.

By definition, $\theta_{x,y} =$ angle between $x, y$ or, $\theta_{x,y} = \arccos \left( \dfrac{\langle x, y \rangle}{|x| \cdot |y|} \right)$

In $\mathbb{R}^2$, $\mathbb{R}^3$, $\theta_{x,y}$ is the "correct angle", [diagram of acute angle between vectors] **Note:** By rearranging, we get a geometric interpretation of

$$\langle x, y \rangle = (\cos \theta_{x,y}) \, |x| \cdot |y| \, , x \neq \mathbf{0}, y \neq \mathbf{0}$$

# 3 Analysis: Convergence of Sequences

We start by reviewing convergence in $\mathbb{R}$:

Let $\{a_\nu\}$ be a sequence in $\mathbb{R}$. The sequence is really just an infinite list of numbers $a_1, a_2, a_3, \ldots$. The greek letter $\nu$ is used so as to avoide using $n, m$, etc. which is usually used as dimensions in Euclidean space—avoid overloading notation.

A more precise notation for a sequence in $\mathbb{R}$ would be:

$$\{a_\nu\}_{v=1}^{\infty}$$

Recall what it means for $\{a_\nu\}$ to converge to some $b \in \mathbb{R}$,

**Notation:** *Converge to* can be written as

$$\lim a_\nu = b \quad \text{or} \quad \lim_{\nu \to \infty} = b \text{ (more precise)} \quad \text{or} \quad a_\nu \to b \text{ (as } \nu \to \infty)$$

**Definition 7.** $a_\nu \to b$ means $\forall \epsilon > 0$, there is $\nu_0 > 0$ such that $\nu > \nu_0 \implies |a_\nu - b| < \epsilon$.

[diagram about epsilon-delta error tolerance]

Here are some important facts about convergence in $\mathbb{R}$,

1) Limit laws: Suppose $\{a_\nu\}, \{b_\nu\}$ are sequences in $\mathbb{R}$ such that $a_\nu \to a$ and $b_\nu \to b$. Then, $a_\nu + b_\nu \to a + b$ and $a_\nu b_\nu \to ab$.
2) Squeeze Theorem (for sequences): $\{a_\nu\}, \{b_\nu\}, \{c_\nu\}$ are sequences in $\mathbb{R}$ and $a_\nu \to l, c_\nu \to l$ and $a_\nu \le b_\nu \le c_\nu$ $\forall \nu$, then $b_\nu \to l$.

**Special Case.** If $|b_\nu| \le c_\nu$ for all $\nu$ and $c_\nu \to 0$, then $b_\nu \to 0$. Also, it goes without saying, $|b_v| \to 0$. This follows because $|b_\nu| \le c_\nu \iff -c_\nu \le b_\nu \le c_\nu$

**Remark.** Sequences are **not** sets. Their elements are ordered.

$$\{a_\nu\}_{\nu=1}^{\infty} = (0, 0, 0, \ldots), \{a_\nu : \nu = 1, 2, 3, \ldots\} = \{0\}$$

The goal for this section is to define convergence of sequences in $\mathbb{R}^n, n \ge 2$

Let $\{x_\nu\}$ be a sequence of vectors in $\mathbb{R}^n$ where 'sequence of vectors' really just means an infinite list of vectors $x_1, x_2, \ldots \in \mathbb{R}^n$

**Notation:** $x_\nu \in \mathbb{R}^n$, to list the coordinates, we write

$$x_\nu = (x_{1,\nu}), x_{2,\nu}, \ldots, x_{n,\nu}$$

We can also use the notation $x_\nu = x^{(\nu)} = \left(x_1^{(\nu)}, x_2^{(\nu)}, \ldots, x_n^{(\nu)}\right)$

**Example 4.** $\left\{\left(\frac{1}{\nu}, 1 + \nu\right)\right\}_{\nu=1}^{\infty}$. As a list:

$$(1, 2), \left(\frac{1}{2}, 3\right), \left(\frac{1}{3}, 4\right), \ldots$$

**Definition 8.** Let $\{x_\nu\}$ be a sequence in $\mathbb{R}^n$, let $p \in \mathbb{R}^n$.

$\{x_\nu\}$ $\underbrace{\text{converges}}_{x_\nu \to \infty}$ to $p$ if the sequence $\{|x_\nu - p|\}$ $\overbrace{\text{converges to}}^{\text{in } \mathbb{R}}$ $0 \in \mathbb{R}$.

In terms of the epsilon definition for a limit, $\{x_\nu\}$ converges to $p$ if for every $\epsilon > 0$, there is $\nu_0$ such that $\nu > \nu_0 \implies |x_\nu - p| < \epsilon$.

What does this mean? It means for every $x, y \in \mathbb{R}^n, |x - y|$ = distance between $x, y$. When $|x_\nu - p|$ is very small, it means $x_\nu$ is very close to $p$.

**Theorem 7.** *Let $\{x_\nu\}$ be a sequence in $\mathbb{R}^n$, let $p \in \mathbb{R}^n$*

$$x_\nu \to p \textit{ if and only if for each } j = 1, 2, \ldots, n, \lim_{\nu \to \infty} x_{j,\nu} \to p_j$$

Each coordinate of $x_\nu$ can be written as

$$x_\nu = \left( \underbrace{x_{1,\nu}}_{\{x_{i,\nu}\}_{\nu=1}^{\infty}}, x_{2,\nu}, \ldots, \underbrace{x_{n,\nu}}_{\{x_{n,\nu}\}_{\nu=1}^{\infty}} \right)$$

**Example 5.** $\left\{\left(\frac{1}{2^v}, e^{\frac{1}{v}}\right)\right\}_{\nu=1}^{\infty}$ (sequence in $\mathbb{R}^2$)

First coordinate sequence converges: $\dfrac{1}{2^v} \to 0$
Second coordinate sequence converges: $e^{\frac{1}{v}} \to e^0 = 1$

Them, by the thm., $\left(\dfrac{1}{2^v}, e^{\frac{1}{v}}\right) \to (0,1)$

**Example 6.** $\left\{\left(\dfrac{1}{v}, 1+v\right)\right\}$ does not converge since the second coordinate sequence $\{1+v\}$ diverges to $+\infty$.

*Proof.* ($\Rightarrow$) Suppose $x_\nu \to p$. Fix some $j = 1, 2, \ldots, n$.
...
($\Leftarrow$) Suppose $x_{j,\nu} \to p_j$ for $j = 1, 2, \ldots, n$ (i.e. $|x_{j,v} - p_j| \to 0$)

We want to show $x_\nu \to p$ (in $\mathbb{R}^n$)
...

$$|x_\nu - p| \leq \sum_{j=1}^{\infty} |x_{j,\nu} - p_j| \to 0$$

$\sum_{j=1}^{\infty} |x_{j,\nu} - p_j|$ is a fine sum of sequence which all $\to 0$. By limit law in $\mathbb{R}$, the right hand side $\to 0$.

By squeeze, $|x_\nu - p| \to 0$. ☐

## 3.1 Componentwise Nature of Convergence

$\big(\text{In } \mathbb{R}^3 \text{ case}\big)$ $\{x_\nu\}$ converges to $p \iff \begin{cases} x_{1,\nu} \to p_1 \\ x_{2,\nu} \to p_2 \\ x_{3,\nu} \to p_3 \end{cases}$

**Theorem 8.** *Linearity of Convergence.*

*(1) If $\{x_\nu\}$, $\{y_\nu\}$ are sequences in $\mathbb{R}^n$ and $x_\nu \to p, y_\nu \to p$, then $x_\nu + y_\nu \to p + q$.*

*(2) If $\{x_\nu\}$ is a sequence in $\mathbb{R}^n$, $x_\nu \to p, c \in \mathbb{R}$, then $cx_\nu \to cp$.*

*Proof.* (1) Suppose $x_\nu \to p, y_\nu \to q$,

By componentwise nature of convergence, we have,

$$x_{j,\nu \to p_j} \text{ and } y_{j,\nu} \to q_j \text{ for } j \in \{1, \ldots, n\}$$

$(x_\nu + y_\nu)_j = x_{j,\nu} + y_{j,\nu} \to p_j + q_j = (p+q)_j$ $\forall j \in \{1, \ldots, n\}$ by limit law in $\mathbb{R}^n$.

By componentwise nature again,

$$x_\nu + y_\nu \to p + q$$

☐

## 3.2 Mappings Between Euclidean Spaces

Note that we don't always require a function or map to be defined on all of a euclidean space.

E.g. $f(x) = \dfrac{1}{x}$ $(x \in \mathbb{R})$

$$\operatorname{dom}(f) = \text{ domain of } f = \{x \in \mathbb{R} : x \neq 0\}$$

$A \subseteq \mathbb{R}^n$ means $A$ is a subset of $\mathbb{R}^n$. $A$ could be equal to $\mathbb{R}^n$, it could possibly be not $\mathbb{R}^n$. If we want $A$ to not be $\mathbb{R}^n$, then we use $A \subsetneq \mathbb{R}^n$ which means $A \subseteq \mathbb{R}^n, A \neq \mathbb{R}^n, A$ is a proper subset.

$A \subseteq \mathbb{R}^n, f : A \to \mathbb{R}^n$ means $f$ is a function with $\operatorname{dom}(f) = A$ and $f(x) \in \mathbb{R}^m$ $\forall x \in A$.

$f(x) \in \mathbb{R}^m$ so it has $m$ coordinates. We write,

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x)).$$

Each $f_j : A \to \mathbb{R}$ is a real-valued function with domain $A$ called $j$-th coordinate function of $f$.

**Example 7.** $f : \mathbb{R} \to \mathbb{R}^2, f(x) = \left(x^2 + 1, 3x\right)$.

Then, $f_1(x) = x^2 + 1, f_2(x) = 3x$.

## 3.3 Function Limits

**Definition 9.** $A \subseteq \mathbb{R}^n, f : A \to \mathbb{R}^m, p \in \mathbb{R}^n, L \in \mathbb{R}^m$

$$\lim_{x \to p} f(x) = L$$

when

(1) There is at least one sequence $\overbrace{\{x_\nu\} \text{ in A}}^{x_\nu \in A \text{ for all } \nu}$ such that $x_\nu \to p$

(2) For all sequences $\{x_\nu\}$ in $A$, $x_\nu \to p$ implies $f(x_\nu) \to L$. e

## 3.4 Point-Set Topology: Compact Sets and Continuity

We will now lay the topological foundations required to defining the derivative. Point-set topology is the study of continuity on spaces. We will only be using a level of topology that is required for this course.

**Definition 10. (ball)** A ball in $\mathbb{R}^n$ is a set of the form

$$\mathcal{B}(p, r) = \{x \in \mathbb{R}^n : |x - p| < r\}$$

for $p \in \mathbb{R}^n, r > 0$ where $p$ is the center of the ball and $r$ is the radius.

We think of balls in euclidean spaces as a region that surrounds a defined point. When we define balls in $\mathbb{R}^n$ for any $n$-dimensions, interesting results follow. In $\mathbb{R}^2$, the 'ball' is really just a circular region centered a point $p$. In $\mathbb{R}^3$, it's a sphere. What would it be in $\mathbb{R}^1$? It's an *open* interval. I have italicized 'open' since this will be defined rigorously later.

It's important to note that the boundary of a ball as defined above is not included in the set since $|x - p| < r$. This follows for all $n$.

### 3.4.1 Bounded Set

**Definition 11. (Bounded Sets)** A set $A \subseteq \mathbb{R}^n$ is *bounded* if there is a ball $\mathcal{B}(p, r)$ such that $A \subseteq \mathcal{B}(p, r)$.

A bounded set can be thought of as a region $A \subseteq \mathbb{R}^n$ that's completely enclosed by a ball $\mathcal{B}(p, r)$ such that all points in $A$ are also included in $\mathcal{B}(p, r)$ but all points in $\mathcal{B}(p, r)$ may or may not be included in $A$.

There are some interesting examples for bounded sets. Consider the following,

(1) All balls are bounded.

(2) For any $p \in \mathbb{R}^n$, any $r > 0$,

$$\mathcal{B}(p, r) = \{x \in \mathbb{R}^n : |x - p| \leq r\}$$

is bounded. Proof by $\overline{\mathcal{B}}(p, r) \subseteq \mathcal{B}(p, r + 1)$.

(3) $\mathbb{R}^n$ is not bounded.

(4) $(0, +\infty)$ is not a bounded subset of $\mathbb{R}$.

(5) A sequence $\{x_\nu\}$ in $\mathbb{R}^n$ is bounded when its set of terms $\{x_1, x_2, \ldots, x_n\}$ is a bounded set.

**Theorem 9.** *If $\{x_\nu\}$ is convergent, then it's bounded.*

*Proof.* Suppose $x_\nu \longrightarrow p$. Recall, this means for every $\epsilon > 0$, $\exists \nu_o$ s.t. $\nu > \nu_0 \implies |x_\nu - p| < \epsilon$. Apply the definition of convergence with $\epsilon = 1$ to get a $\nu_o$ s.t. $\nu > \nu_0 \implies |x_\nu - p| < 1$.

Let $M = \max\left(\{|x_\nu - p| : \nu = 1, 2, \ldots, \nu_0\} \cup \{1\}\right)$

Then, simple to check that

$$\{x_1, x_2, \ldots, x_{\nu_0}, x_{\nu_0+1,\ldots}\} \subseteq \mathcal{B}(p, m + 1)$$

$\square$

### 3.4.2 Closed Set

**Definition 12. (Closet Sets)** A set $F \subseteq \mathbb{R}^n$ is *closed* if for every sequence $\{x_\nu\}$ in $F$ which happens to converge in $\mathbb{R}^n$, in fact converges in $F$, that is, $\lim_{\nu \to \infty} x_\nu \in F$.

It can also be said that $F$ is a closed set if $F$ containts all of its limit points. Let's consider some examples of closed sets.

(1) Closed intervals in $\mathbb{R}$ are closed such as $[1, 2]$. $[1, +\infty)$ is also a closed set since its endpoints are real numbers.

(2) $(0, 1]$ is not closed. *Proof.* Consider the sequence $\left\{\frac{1}{\nu}\right\}$ in $A$ which converges to 0 but $\lim \frac{1}{\nu} \neq 0 \in A$.

(3) Let $n = 1, 2, \ldots$. $S^{n-1} \subseteq \mathbb{R}^n$ , called the $(n-1)$th dimensional sphere is defined by

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$$

(4) $\mathbb{R}^n$ is a closed set.

Example (3) is important. Intuitively, $S^0$ is two points $-1$ and 1; $S^1$ is a unit circle and $S^2$ is a unit sphere.

**Claim.** For each $n = 1, 2, \ldots$, $S^{n-1}$ is closed.

*Proof.* Let $\{x_\nu\}$ be a sequence in $S^{n-1}$, and suppose $x_\nu \longrightarrow p$. We want to show $p \in S^{n-1}$, i.e., $|p| = 1$.

Since $|\cdot|$ is continuous, $|x_\nu| \longrightarrow |p|$. But $\{|x_\nu|\}$ is a constant sequence with value 1, hence $1 \longrightarrow |p| \implies |p| = 1$. $\square$

A similar proof show that for any $p \in \mathbb{R}^n$ and $r > 0$,

$$\overline{\mathcal{B}}\left(p, r\right) = \{x \in \mathbb{R}^n : |x - p| \leq r\}$$

is closed (we call it a closed ball).

### 3.4.3 Compact Set

**Definition 13. (Compact Sets)** A set $K \subseteq \mathbb{R}^n$ is *compact* if it is both *closed* and *bounded.*

Following this definition, we will understand the Extreme Value Theorem in the context of several variables. Consider the following examples,

(1) $S^{n-1}$, $n = 1, 2, \ldots$ are all compact.

(2) Bounded, closed intervals $[a, b]$ for $(a, b) \in \mathbb{R}$ are compact.

(3) $R^n$ is closed but not bounded, so it's not compact.

**Theorem 10.** ***(Extreme Value Theorem)*** *Let $K \subseteq A \subseteq \mathbb{R}^n, K \neq \varnothing$ and $K$ is compact. $f\colon A \longrightarrow \mathbb{R}$ (real valued) continuous on $K$, then, $\exists x_{max}, x_{min}$ s.t.*

$$f\left(x_{min}\right) \leq f\left(x\right) \leq f\left(x_{max}\right)$$

*for any $x \in K$.*

**Remark.** The single-variable version of the extreme value theorem suggests $K = [a, b]$ which is closed, bounded. Using single-variable functions, we can see that $K$ needs to be both closed and bounded (i.e., compact).

Let's explore an application of the Extreme Value Theorem. Let $K \subseteq \mathbb{R}^n$ be non-empty and compact. Then, there are vectors in $K$ which are longest and shortest with respect to vectors in $K$.

*Proof.* $|\cdot|$ is a continuous function. By extreme value theorem (EVT) applied to $|\cdot|$ on $K$, there are $x_{\max}, x_{\min} \in K$ such that $|x_{\min}| \leq |x| \leq |x_{\max}|$ for all $x \in K$. $\square$

# 4 Linear Algebra

We will now explore the idea behind linear transformations and move towards defining the derivative in several variables.

## 4.1 Matrices

**Definition 14. (Matrix)** A matrix $A$ is a rectangular array of numbers:

$$A = \left[a_{ij}\right]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where $m$ is the number of rows and $n$ is the number of columns.

Consider the example,

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 6 \end{bmatrix}$$

is a $2 \times 3$ matrix.

Note that it's somewhat tedious to manually write matrix entries both by hand and on LaTeX and so it's preferable to refer to some matrix $A$ as just $[a_{ij}]_{m\times n}$.

**Remark.** When all the entries of $A$ are real, then $A$ is called a real matrix.

**Example 8.** The $m \times n$ matrix $E_{i_0 j_0}$ $(1 \le i_0 \le m, 1 \le j_0 \le n)$ is

$$E_{ij} = \begin{bmatrix} \delta_{ii_0} & \delta_{jj_0} \end{bmatrix}_{m\times n}$$

where $\delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise} \end{cases}$

The $3 \times 3$ matrix,

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

### 4.1.1 Operations with Matrices

Matrix addition is entrywise.

$$A + B = \begin{bmatrix} a_{ij} \end{bmatrix}_{m\times n} + \begin{bmatrix} b_{ij} \end{bmatrix}_{m\times n} = \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}_{m\times n}$$

Scalar multiplication of matrices is entrywise.

$$cA = c\begin{bmatrix} a_{ij} \end{bmatrix}_{m\times n} = \begin{bmatrix} ca_{ij} \end{bmatrix}_{m\times n}$$

**Example 9.**

$$2\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & 10 \end{bmatrix}$$

**Remark.** Matrix Addition and Scalar Multiplication of Matrices have all the same nice algebraic properties as vector addition and scalar multiplication. E.g., $A + B = B + A, (A + B) + C = A + (B + C), c(dA) = (cd)A$, etc.

### 4.1.2 Matrix by Vector Multiplication

If $A$ is a $m \times n$ matrix and $x$ is an $n$-dimensional (column) vector,

$$x = (x_1, \ldots, x_n) \implies \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

We can think of Matrix by Vector multiplication in multiple ways,

(1) By column-wise approach,

$$A_{m\times n} = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$$

where $v_1, \dots, v_n$ are the $m$-dimensional columns of $A$.

$$Ax = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$

This approach of Matrix by Vector multiplication is nice theoretically.

(2) By row-wise approach,

$$A_{m \times n} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

where $w_1, \dots, w_m$ are the $n$-dimensional rows of $A$.

$$Ax = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \langle w_1, x \rangle \\ \vdots \\ \langle w_n, x \rangle \end{bmatrix}$$

This approach of Matrix by Vector multiplication is nice computationally.

**Example 10.**

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + (-1) \cdot 4 \\ 2 \cdot 2 + 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2 + \begin{bmatrix} -1 \\ 3 \end{bmatrix} 4$$

We can generalize Matrix by Vector multiplication by a nice formula,

$$Ax = \left[a_{ij}\right]_{m \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix}$$

**Remark.** $Ax$ is a $m$-dimensional column vector. If $A$ is $m \times n$ and $x$ is $n \times 1$, then we can think of cancelling the common $n$ and the resultant vector is $m \times 1 = m$.

## 4.2 Linear Mappings: Linear Transformations

**Proposition.** Matrix by Vector multiplication is linear, i.e., if $A$ is $m \times n$ matrix, then,

(1) $\forall x, y \in \mathbb{R}^n, A(x + y) = Ax + Ay$

(2) $\forall c \in \mathbb{R}, x \in \mathbb{R}^n, A(cx) = c(Ax)$

**Definition 15. (Linear Transformation)** A *linear transformation* is a map $T \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, such that,

(1) For every $x, y \in \mathbb{R}^n, T(x + y) = T(x) + T(y)$

(2) For every $c \in \mathbb{R}$, $x \in \mathbb{R}^n$, $T(cx) = cT(x)$

When we say that $T \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear, it means that $T$ is a linear transformation. Under this definition, we note that for $\mathbb{R}^n, \mathbb{R}^m$, the zero map $T_0 \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $T_0(x) = 0 \; \forall x \in \mathbb{R}^n$ is a linear transformation.

A real-valued linear transformation is such that for any $y \in \mathbb{R}^n$, the function $T_y \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

$$T_y(x) = \langle x, y \rangle$$

is linear. *Proof* by properties of $\langle \cdot, \cdot \rangle$.

In fact, functions of this form are the only real-valued linear transformations.

*Proof.* Suppose $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is linear. Define $y = (T(e_1), T(e_2), \ldots, T(e_n)) \in \mathbb{R}^n$ Then, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} T(x) &= T(x_1 e_1 + \ldots + x_n) \\ &= x_1 T(e_1) + \ldots + x_n T(e_n) && \text{(By Linearity of } T\text{)} \\ &= \langle x, y \rangle = T_y(x) \implies T = T_y \end{aligned}$$

□

Linear transformations are closely related to matrices. For a $m \times n$ matrix $A$, $T_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by

$$T_A(x) = Ax$$

is linear (by properties of matrix by vector multiplication). In fact, every linear $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of the form $T(x) = Ax$ for some $m \times n$ matrix $A$. This matrix is unique and is called **the matrix of** $T$.

*Proof.* Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear, set

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \ldots & T(e_n) \end{bmatrix}$$

Then, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} T(x) &= T(x_1 e_1 + \cdots + x_n e_n) \\ &= x_1 T(e_1) + \cdots + x_n T(e_n) \\ &= \begin{bmatrix} T(e_1) & \ldots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax \end{aligned}$$

□

Let's see a linear transformation in action.

**Example 11.** Rotation about the origin by $\frac{\pi}{2}$ radians counterclockwise in $\mathbb{R}^2$ is a linear transformation. Let $T_{\frac{\pi}{2}}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be linear. Its matrix is

$$A = \begin{bmatrix} T_{\frac{\pi}{2}}(e_1) & T_{\frac{\pi}{2}}(e_2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We can conclude that every linear transformation $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is represented by a unique matrix $A$ in the sense that $T(x) = Ax \forall x \in \mathbb{R}^n$. This unique matrix $A$

$$A = \begin{bmatrix} T_1(e_1) & \ldots T_n(e_n) \end{bmatrix}$$

by applying $T$ to all the standard basis vectors in order. $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$.

**Example 12.** $T_0\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the zero transformation. Then, the matrix representation of $T_0$ is $[0]_{m \times n}$.

**Example 13.** $\mathrm{id}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map on $\mathbb{R}^n$, $\mathrm{id}\,(x) = x$. id is linear and its matrix representation is,

$$A = \begin{bmatrix} \mathrm{id}\,(e_1) & \dots & \mathrm{id}\,(e_n) \end{bmatrix} = \begin{bmatrix} e_1 & \dots & e_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \delta_{ij} \end{bmatrix}_{m\times n} = \mathrm{I}_n$$

**Example 14.** Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a reflection across the line $y = x$ in $\mathbb{R}^2$. Its matrix representation is

$$A = \begin{bmatrix} T\,(e_1) & T\,(e_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

**Theorem 11.** *Every linear transformation is continuous on its domain.*

*Proof.* Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear, let $A = [v_1 \cdots v_n]$ be the matrix of $T$. For any $x \in \mathbb{R}^n$,

$$T\,(x) = Ax = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + \cdots + x_n v_n$$

So $T$ is obtained from a composition of scalar multiplication and vector addition. Then, $T$ is continuous since composition of continuous functions. $\square$

**Theorem 12.** *Let $T, S\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear, let $c \in \mathbb{R}$. Then, $T + S$ and $cT$ are also linear.*

The proof is mostly skipped, however, it can be noted that,

$$\begin{aligned} (T+S)\,(x+y) &= T\,(x+y) + S\,(x+y) \\ &= (T\,(x) + T\,(y)) + (S\,(x) + S\,(y)) \\ &= (T\,(x) + S\,(x)) + (T\,(y) + S\,(y)) \\ &= (T+S)\,(x) + (T+S)\,(y) \end{aligned}$$

**Remark.** The matrix of $T$ is $A$ and the matrix of $S$ is $B$, then, it follows that the matrix of $T + S$ is $A + B$ and the matrix of $cT$ is $cA$.

### 4.2.1 Matrix by Matrix Multiplication

To mulyiply two matrices $A$ and $B$, we need the dimenstions to satisfy

$$\underbrace{A}_{m\times n}\underbrace{B}_{n\times l} = \underbrace{AB}_{m\times l}$$

**Definition 16.** $A$ is $m \times n$, $B$ is $n \times l$, where

$$B = \begin{bmatrix} v_1 & \cdots & v_l \end{bmatrix}$$

then,
$$AB = A \begin{bmatrix} v_1 & \cdots & v_l \end{bmatrix} = \begin{bmatrix} Av_1 & \cdots & Av_l \end{bmatrix}$$
where $v_1, \dots, v_l$ each are an $n$-dimensional column vector.

There is a nice formula that expresses the $i, j$ coordinate as,

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \langle i^{\text{th}} \text{ row of } A, j^{\text{th}} \text{ column of } B\rangle$$

**Example 15.**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 1 \\ -4 & 10 & 3 \end{bmatrix}$$

Assuming the dimensions make sense, consider the following properties for matrices,

(1) $A(BC) = (AB)C$

(2) $A(B+C) = AB + AC$

(3) $(A+B)C = AC + BC$

It seems tempting to also say that matrix multiplication is commutative since it looks very similar to the properties of vector spaces, however, matrix multiplication is **not** commutative. That is to say,

$$AB \neq BA$$

**Example 16.** $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then,

$$AB = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

It follows that $AB \neq BA$.

**Theorem 13.** *$T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m, S\colon \mathbb{R}^m \longrightarrow \mathbb{R}^l$ both $T, S$ linear. Let $A$ be matrix of $T$, let $B$ be matrix of $S$. Then, the matrix of $S \circ T$ is $\underbrace{B}_{l\times m}\underbrace{A}_{m\times n} = \underbrace{BA}_{l\times n}$ (makes sense because $S \circ T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^l$)*

*Proof.* Trivial.

$$\begin{aligned} (S \circ T)(x) &= S(T(x)) \\ &= S(Ax) \\ &= B(Ax) \\ &= (BA)x \end{aligned}$$

□

**Definition 17. (Matrix Inverses)** An $n \times n$ matrix $A$ is invertible if there exists a $n \times n$ matrix $B$ such that

$$AB = I_n = BA$$

The notation is $B = A^{-1}$ that is to say, when $B = A^{-1}$, then $A = B^{-1}$.

**Remark.** It does not make sense to talk about a non-square matrix being invertible. If $A$ is not square, then there is no $B$ such that $AB$ and $BA$ both defined.

## 4.3 Matrix Norms

Let $A$ be an $m \times n$ matrix, let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T(x) = Ax$. Recall that $|T(x)|$ is continuous and $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. By the *extreme value theorem*, $\exists x_0 \in S^{n-1}$ s.t.

$$|Ax| = |T(x)| = \max\left\{|T(x)| : x \in S^{n-1}\right\} = \max\left\{|Ax| : x \in S^{n-1}\right\}$$

We define the norm of $A$ to be

$$\|A\| = \max\left\{|Ax| : x \in S^{n-1}\right\}$$

For the linear transformation $T(x) = Ax$, this quantity is also defined to be the norm of

$$\|T\| = \|A\| = \max\left\{|T(x) : x \in S^{n-1}\right\}$$

The following are examples of matrix norms,

(1) $T_0 \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is zero transformation, $\|T\| = 0$ since $|T(x)| = 0\ \forall x \in \mathbb{R}^n$. Similarly, $\|0_{m\times n}\| = 0$

(2) Let $R_\theta \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be counterclockwise rotation by $\theta$. Then $\|R_\theta\| = 1$ since $|R_\theta(x)| = |x| = 1\ \forall x \in S^{n-1}$

In general, to compute $\|T\|$, $\|A\|$, we have to solve a multivariable optimization problem. We're not interested in that, we're interested in the theoretical applications of matrix norms, for example,

**Theorem 14.** *Let $A$ be an $m \times n$ matrix. Then, for any $x \in \mathbb{R}^n$, we have*

$$|Ax| \le \|A\| \cdot |x|$$

*(also, if $T \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear, then $|T(x)| \le \|T\|\,|x|\ \forall x \in \mathbb{R}^n$)*

*Proof.* Let $x \in \mathbb{R}^n$. When $x = 0$, result is trivial, so we may assume $x \neq 0$, i.e., $|x| > 0$.

Then, we have $\frac{1}{|x|}x \in S^{n-1}$ since $\left|\frac{1}{|x|}\right| = \frac{1}{|x|}|x| = 1$.

Then,

$$|Ax| = \left|A\left(\frac{|x|}{|x|}x\right)\right| = |x|\left|A\left(\frac{x}{|x|}\right)\right| \le |x|\,\|A\|$$

by the definition $\|A\| = \max\left\{|Ay| \colon y \in S^{n-1}\right\}$ □

**Proposition.** Let $A$ be a $m \times n$ matrix. Then,

$$\|A\| = 0 \iff A = 0_{m\times n}$$

*Proof.* ( $\Longleftarrow$ ) trivial.

( $\Longrightarrow$ ) Suppose $\|A\| = 0$. By the theorem, we have $|Ax| \le \|A\| \cdot |x| = 0\ \forall x \in \mathbb{R}^n$. Thus, $|Ax| = 0$, hence $Ax = 0\ \forall x \in \mathbb{R}^n$.

Then, $A = 0_{m\times n}$. □

## 4.4 Towards Defining the Derivative

In single-variable calculus, $f \colon \mathbb{R} \longrightarrow \mathbb{R}, a \in \mathbb{R}$,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

when the limit actually exists. We will now rewrite $f'(a)$ as

$$0 = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} - f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a) - f'(a)h}{h}$$

We say that $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \rightsquigarrow f'(a) \approx \frac{f(a+h) - f(a)}{h}$ when $h$ is small enough and $\rightsquigarrow f(a+h) \approx f(a) + f'(a)h$ which is the linear approximation of $f$ near $a$.

We introduce an error term $\varepsilon(h) = f(a+h) - (f(a) + f'(a)h)$. It's intuitive to see that

$$\varepsilon(h) \longrightarrow 0 \text{ as } h \longrightarrow 0$$

But this is not a strong enough statement. A stronger statement is to say that

$$\lim_{h\to 0} \frac{\varepsilon(h)}{h} = \lim_{h\to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0$$

This says that not only is $\varepsilon(h) \longrightarrow 0$ as $h \longrightarrow 0$ but in fact $\varepsilon(h)$ is going to 0 much faster than $h$ goes to 0. In general, what we're trying to achieve is to minimize a bounded area between the function and the derivative at that point which is $\varepsilon(h)$.

### 4.4.1 Generalizing to Several Variables

In single-variable, $T_a(h) = f'(a)h$ is a linear transformation $\mathbb{R} \longrightarrow \mathbb{R}$. So, for multivariable functions, the derivative at a vector will not be a vector, but a linear transformation

$$Df_a \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

where $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$. The matrix of $Df_a$ is called a **Jacobian Matrix**.

We also have to be careful in saying "goes to 0 faster than $h \longrightarrow 0$". Let $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

$$\lim_{h\to 0_n} \frac{|f(a+h) - f(a) - Df_a(h)|}{|h|} = 0$$

A nice way to think about this is,

$$f(a+h) = f(a) + Df_a(h) + o(h)$$

where instead of $\varepsilon(h)$, we write $o(h)$ (*little oh of* $h$) to express the error terms where $\dfrac{|o(h)|}{|h|} \longrightarrow 0$ as $h \longrightarrow 0$. We will study the behavior of functions through $o(h)$ and other forms that also trace functional behavior through the *Bachmann-Landau Notation* in the next section.

# 5 Derivatives in Several Variables

## 5.1 Bachmann-Landau Notation

**Definition 18.** Consider $\varphi \colon \mathcal{B}(0_n, \varepsilon) \longrightarrow \mathbb{R}^m$

(1) $\varphi$ is $o(1)$ ("little oh of 1" or "smaller than order 1") if for every $c > 0$, $|\varphi(h)| \le c$ for all small enough $h$.

$|\varphi(h)| \le c$ is an abbreviation for 'there is $\delta > 0$ s,t, $|h| < \delta \implies |\varphi(h)| \le c$'. Note that $\delta$ depends on $c$. For example, a smaller $c$ will require a smaller $\delta$, generally. Equivalently, $\varphi(h) \longrightarrow 0$ as $h \longrightarrow 0$.

(2) $\varphi$ is $o(h)$ ("little oh of $h$" or "smaller than order $h$") if for every $c > 0$, $|\varphi(c)| \le c|h|$ for all small enough $h \implies \frac{|\varphi(h)|}{|h|} \le c$.

Equivalently, $\lim_{h\longrightarrow 0} \frac{|\varphi(h)|}{|h|} = 0$ and $\varphi(h) = 0$. The idea is that $\varphi$ is $o(h)$ means $\varphi$ goes to 0 faster than $h$ goes to 0.

(i) (3) $\varphi$ is $O(h)$ ("big oh of $h$" or "of order $h$") *if there is a* $c > 0$ such that $|\varphi(h)| \le c|h|$ for all small enough $h$.

This is a much weaker claim than $o(h)$ since it asks for only *one instance* but $o(h)$ asks *for every* $c$. Eauivalently, there is a bound on $\frac{|\varphi(h)|}{|h|}$ for all small enough $h$. The idea is that $\varphi$ is $O(h)$ means $\varphi$ goes to 0 *at least* as fast as $h$ goes to 0.

**Proposition.** $o(h) \subsetneq O(h) \subsetneq o(1)$. $\varphi$ is $o(h) \implies \varphi$ is $O(h) \implies \varphi$ is $o(1)$.

**Theorem 15.** *Every linear* $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is* $O(h)$.

*Proof.* Set $c = \|T\|$. Then,

$$|T(h)| \le \|T\| \cdot |h| = c|h| \forall h \in \mathbb{R}^n$$

□

**Theorem 16.** *A linear* $T$ *is* $o(h) \iff T$ *is the zero transformation.*

*Proof.* ( $\Longleftarrow$ ) trivial.

( $\implies$ ) Suppose $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear and $o(h)$. $\|T\| = \max\left\{|T(x)|\colon x \in S^{n-1}\right\}$, so by Extreme Value Theorem, $\exists x_0 \in S^{n-1}$ s.t. $\|T\| = |T(x_0)|$.

Set $h = sx_0, s \in (0,1)$.

$$\begin{aligned} 0 &= \lim_{h\to 0} \frac{|T(h)|}{|h|} = \lim_{s\to 0} \frac{|T(sx_0)|}{|sx_0|} = \lim_{s\to 0} \frac{s|T(x_0)|}{s} \\ &= \lim_{s\to 0} |T(x_0)| = |T(x_0)| = \|T\| \end{aligned}$$

$\implies 0 = \|T\| \implies T$ is the zero transformation. □

Keep in mind the notation. $o(h)$ is often used to denote some unspecified $o(h)$ function. For example, $f(h) = g(h) + o(h) \iff f(h) - g(h) = o(h) \iff f(h) - g(h)$ is $o(h)$.

**Proposition.** $o(h) + o(h) = o(h)$.

In other words, the sum of any two $o(h)$ functions is also $o(h)$. We will follow this with saying, $c \cdot o(h) = o(h)$ for any $c \in \mathbb{R}$, that is to say, the persistence of $o(h)$ under scalar multiplication.

*Proof.* $o(h) + o(h) = o(h)$

Let $\varphi, \psi\colon \mathcal{B}(0_n, \varepsilon) \longrightarrow \mathbb{R}^m$ both be $o(h)$. Then,

$$\lim_{h\to 0} \frac{|\varphi(h) + \psi(h)|}{|h|} \le \lim_{h\to 0} \frac{|\varphi(h)| + |\psi(h)|}{|h|} = \lim_{h\to 0} \frac{|\varphi(h)|}{|h|} + \lim_{h\to 0} \frac{|\psi(h)|}{|h|} = 0 + 0 = 0$$

Also, $(\varphi + \psi)(0) = \varphi(0) + \psi(0) = 0 + 0 = 0$.

Thus, $\varphi + \psi$ is $o(h)$. □

**Proposition.** $o(1) + o(1) = o(1), c \cdot o(1) = o(1)$ and $O(h) + O(h) = O(h), c \cdot O(h) = O(h)$ for any $c \in \mathbb{R}$.

**Proposition.** $o(1) \cdot O(h) = o(h)$. i.e., if $\varphi$ is $o(1)$, $\psi$ is $O(h)$, then, $|\varphi(h)| \cdot |\psi(h)|$ is $o(h)$.

The proof will not be stated, however, the sketch of the proof is,

$$\lim_{h\to 0} \frac{|\varphi(h)| \cdot |\psi(h)|}{|h|} = \lim_{h\to 0} |\varphi(h)| \left( \frac{|\psi(h)|}{|h|} \right)$$

$\frac{|\psi(h)|}{|h|}$ is bounded since $\psi$ is $O(h)$. It follows that,

$$\lim_{h\to 0} |\varphi(h)| \left( \frac{|\psi(h)|}{|h|} \right) \leq \lim_{h\to 0} |\varphi(h)| \cdot M = M \lim_{h\to 0} |\varphi(h)| = M \cdot 0 = 0$$

Note that $\varphi(h)$ is $o(1)$.

**Corollary.** $o(h) \cdot o(h) = o(h)$

## 5.2 Defining the Derivative

Recall that

$$\begin{cases} |o(1)| \longrightarrow 0 \\ \dfrac{|O(h)|}{|h|} \text{ is bounded} \\ \dfrac{|o(h)|}{|h|} \to 0 \end{cases}$$

as $h \to 0$ and $o(h) \subsetneq O(h) \subsetneq o(1)$. Furthermore, $o(h) + o(h) = o(h), O(h) + O(h) = O(h)$, all linear transformations are $O(h)$ and the only $o(h)$ linear transformation is the zero map.

**Definition 19. (Interior point)** Let $A \subseteq \mathbb{R}^n$, then $a \in A$ is an *interior point* of $A$ if $\exists \varepsilon > 0$ s.t. $\mathcal{B}(a, \varepsilon) \subseteq A$.

**Remark.** Points on the boundary of $A$ are *not* interior points.

**Definition 20.** Let $A \subseteq \mathbb{R}^n$, $a \in A$ be an interior point. $f \colon A \longrightarrow \mathbb{R}^m$ is differentiable at $a$ if there is a linear transformation $T_a \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t.

$$f(a+h) - f(a) - T_a(h) = o(h)$$

**Remark.** $f(a+h) = f(a) + T_a(h) + o(h)$. $T_a$ is a linear approximation of $f$ near $a$ whose error term is $o(h)$.

**Proposition.** If $f$ is differentiable at $a$, then $T_a$ is *unique.*

We will now use the notation $T_a = Df_a$. The matrix of $Df_a$ is denoted $f'(a)$ and is called the *Jacobian matrix* of $f$ at a.

$$Df_a = f'(a)\, h$$

**Remark.** This proposition means that when $Df_a$ exists, it is the single best linear approximation.

*Proof.* Suppose $T_a, S_a \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ both satisfy the definition of derivative. We want to show $T_a = S_a$. It suffices to show $T_a - S_a$ is the zero transformation.

$$\begin{aligned} T_a(h) - S_a(h) &= (f(a+h) - f(a) + o(h)) - (f(a+h) - f(a) + o(h)) \\ &= o(h) + o(h) = o(h) \end{aligned}$$

The only $o(h)$ linear transformation is the zero mao. Therefore, $T_a - S_a$ is the zero transformation $\implies T_a = S_a$. ☐

We must be careful with the Bachmann-Landau notation and avoid writing $o(h) - o(h) = 0$ since $o(h)$ is not the typical kind of function we deal with, it's simply a general statement regarding the error terms for a function.

**Example 17.** Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear. Then, for any $a \in \mathbb{R}^n, T$ is differentiable at $a$ and $DT_a = T$.

*Proof.* We just need to check $T(a+h) - T(a) - DT_a(h) = o(h)$.

$$\begin{aligned} T(a+h) &= T(a+h) - T(a) - T(h) \\ &= T(a) + T(h) - T(a) - T(h) \\ &= 0 = o(h) \end{aligned}$$

The zero function is $o(h)$. ☐

**Example 18.** Constant maps are differentiable at every interior point of their domain and their derivative is the zero transformation.

*Proof.* Trivial. ☐

**Example 19.** $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, f(x,y) = x^2 + y^2$. $f$ is differentiable at every $(a,b) \in \mathbb{R}^2$ and

$$Df_{(a,b)}(h,k) = 2ah + 2bk$$

where $(h,k) \in \mathbb{R}^2$.

*Proof.* Easy to check that $T(h,k) = 2ah + 2bk$ is linear. It's enough to show that $f(a+h, b+h) - f(a,b) - T(h,k)$ is $o(h,k)$.

$$\begin{aligned} f(a+h, b+h) &= (a+h)^2 + (b+k)^2 - \left(a^2 + b^2\right) - (2ah + 2bk) \\ &= a^2 + 2ah + h^2 + b^2 + 2bh + h^2 - a^2 - b^2 - 2ah - 2bk \\ &= h^2 + k^2 = |(h,k)|^2 \end{aligned}$$

It's easy to check that $|(h,k)|^2$ is $o(h,k)$ since $\lim \frac{|(h,k)|^2}{|(h,k)|} = 0$ ☐

The *Jacobian matrix* at $(a,b)$ where $Df_{(a,b)}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is $f'(a,b) = \begin{bmatrix} 2a & 2b \end{bmatrix}$

**Theorem 17.** *$f\colon A \longrightarrow \mathbb{R}^m$, $A \subseteq \mathbb{R}^n, a \in A$ interior point. If $f$ is differentiable at $a$, then $f$ is continuous at $a$.*

*Proof.*

$$\begin{aligned} f(a+h) - f(a) &= (f(a+h) - f(a) - Df_a(h)) + Df_a(h) \\ &= O(h) + O(h) \\ &= O(h) = o(1) \end{aligned}$$

Thus, $f(a+h) - f(a) \longrightarrow 0$ as $h \longrightarrow 0$, hence, $f(a+h) \longrightarrow f(a)$ as $h \longrightarrow 0$, so $f$ is continuous at $a$. ☐

**Theorem 18.** ***(Linearity of Differentiation)*** *$A \subseteq \mathbb{R}^n, f\colon A \longrightarrow \mathbb{R}^m, a \in A$ interior point. If $f, g$ both differentiable at $a$, then,*

*(1) $f+g$ is differentiable at $a$ and $D\left(f+g\right)_a = Df_a + Dg_a$*

*(2) For any $c \in \mathbb{R}$, $cf$ is differentiable at $a$ and $D\left(cf\right)_a = c\left(Df_a\right)$*

*Proof.* (1) $T\left(h\right) = Df_a\left(h\right) + Dg_a\left(h\right)$ is linear since it is the sum of two linear maps.

$$
\begin{aligned}
&\left(f+g\right)\left(a+h\right) - \left(f+g\right)\left(a\right) - T\left(h\right) \\
&\qquad = f\left(a+h\right) - g\left(a+h\right) - \left(f\left(a\right) + g\left(a\right)\right) - \left(Df_a\left(h\right) + Dg_a\left(h\right)\right) \\
&\qquad = \left(f\left(a+h\right) - f\left(a\right) - Df_a\left(h\right)\right) + \left(g\left(a+h\right) - g\left(a\right) - Dg_a\left(h\right)\right)
\end{aligned}
$$

By definition of the derivative,

$$= o\left(h\right) + o\left(h\right) = o\left(h\right)$$

□

## 5.3 Derivative Rules

The goal of this subsection is to derive the chain rule using the *Bachmann-Landau Notation* and subsequently, derive the quotient/product rules for serveral variables. In single-variable calculus, it's common to see the chain rule to be introduced after the quotient/product rule, however, will see that we require the chain rule to derive the product rule.

*Proof.* $o\left(O\left(h\right)\right) = o\left(h\right)$

Suppose $\psi\left(h\right)$ is $O\left(h\right)$, $\varphi\left(k\right)$ is $o\left(k\right)$, $\psi \circ \varphi$ makes sense (i.e., composable) defined on some $\mathcal{B}\left(0_n, \varepsilon\right)$. We want to show that $\varphi\left(\psi\left(k\right)\right)$ is $o\left(h\right)$.

We know that $|\psi\left(h\right)| \leq d|h|$ since $\psi$ is $O\left(h\right)$. Let $c > 0, |\varphi\underbrace{\left(\psi\left(h\right)\right)}_{k}| \leq a|\underbrace{\psi\left(h\right)}_{k}| = ad|h|$.

We want to show that $\exists \delta > 0$ s.t. $|h| < \delta \implies |\varphi\left(\psi\left(h\right)\right) \leq c|h|$.

Since $\psi\left(h\right)$ is $O\left(h\right)$, $\exists d > 0$ and $\delta_1 > 0$ s.t. $|h| < \delta_1 \implies |\psi\left(h\right)| \leq d|h|$.

Since $\varphi\left(k\right)$ is $o\left(k\right)$, $\exists \delta_2 > 0$ s.t. $|k| < \delta_2 \implies |\varphi\left(k\right)| \leq \dfrac{c}{d}|k|$.

Since $\psi\left(h\right)$ is also $o\left(1\right)$, $\exists \delta_3 > 0$ s.t. $|h| < \delta_3 \implies |\psi\left(h\right)| < \delta_2$.

Set $\delta = \min\left(\delta_1, \delta_3\right)$, then, for any $h$ with $|h| < \delta$,

$$|\varphi\left(\psi\left(h\right)\right)| \underbrace{\leq}_{k=\psi(h), |h|<\delta_3 \implies |k|=|\psi(h|<\delta_2)} \frac{c}{d}|\psi\left(h\right)| \overbrace{\leq}^{|h|<\delta_1} \frac{c}{d} \cdot d|h| = c|h|$$

□

**Theorem 19.** ***(Chain Rule.)*** *$A \subseteq \mathbb{R}^n, B \subseteq \mathbb{R}^m, f\colon A \longrightarrow \mathbb{R}^m, f\left(A\right) \subseteq B, g\colon B \longrightarrow \mathbb{R}^l$ (so $g \circ f$ makes sense), $a \in A$ interior point of $A$, $f\left(a\right) \in B$ interior point of $B$. If $f$ is differentiable at $a$ and $g$ is differentiable at $f\left(a\right)$, then $g \circ f$ is differentiable at $a$, and*

$$D\left(g \circ f\right)_a = Dg_{f(a)} \circ Df_a$$

Note that this is matrix multiplication so the order matters.

*Proof.* Clearly, $Dg_{f(a)} \circ Df_a$ is linear (by composition of linear transformations is a linear transformation). It follows that,

$$\begin{aligned}(g \circ f)(a+h) = g(f(a+h)) = g\left(f(a) + \underbrace{Df_a(h) + o(h)}_{k}\right) \\ &= g(f(a)) + Dg_{f(a)}(Df_a(h) + o(h)) + o(Df_a(h) + o(h)) \\ &= (g \circ f)(a) + Dg_{f(a)}\left(Df'_a(h)\right) + \underbrace{Dg_{f(a)}(o(h)) + o(Df_a(h) + o(h))}_{\text{show that this is } o(h)}\end{aligned}$$

$Dg_{f(a)}(o(h))$ is simply $O(o(h))$. Thus, it follows that,

$$\begin{aligned}Dg_{f(a)}(o(h)) + o(Df_a(h) + o(h)) &= O(o(h)) + o(O(h) + o(h)) && \text{(Lin. Tra. are } O(h)) \\ &= o(h) + o(O(h)) && (O(o(h)) = o(h)) \\ &= o(h) + o(h) = o(h) && (o(O(h)) = o(h))\end{aligned}$$

□

**Lemma.** It follows,

(1) $p\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, p(x,y) = xy$ is differentiable on $\mathbb{R}^2$ and $Dp_{(a,b)}(h,k) = bh + ak, (h,k) \in \mathbb{R}^2$.

(2) $r\colon \mathbb{R}-\{0\} \longrightarrow \mathbb{R}, r(x) = \dfrac{1}{x}$ is differentiable on its domain, $r'(a) = \dfrac{-1}{a^2}, Dr_a(h) = \dfrac{-h}{a^2}$.

*Proof.* (1) $p(a+h, b+k) - p(a,b) - (bh + ak)$. We want to show that the whole expression is $o((h,k))$.

$$\begin{aligned}&= (a+h)(b+k) - abh - bh - ak \\ &= ab + ak + bh + hk - ab - bh - ak \\ &= hk\end{aligned}$$

Claim that $hk$ is $o((h,k))$.

$$|hk| = |h||k| \leq |(h,k)| \cdot |(h,k)| = \underbrace{|(h,k)|}_{o(h,k)}{}^2$$

So, done by a simple argument using the next lemma. □

**Lemma.** $(|\varphi(h)| \leq o(h) \,\forall$ small enough $h) \implies \varphi(h)$ is $o(h)$. Same for $O(h), o(1)$.

*Proof.* Simple. □

**Theorem 20.** *Let $A \subseteq \mathbb{R}^n, f, g\colon A \longrightarrow \mathbb{R}$, both differentiable at an interior point $a \in A$.*

*(1) $fg\colon A \longrightarrow \mathbb{R}$ is differentiable at $a$ and $D(fg)_a = g(a) \cdot Df_a + f(a) \cdot Dg_a$*

*(2) If $g(a) \neq 0$, then $f/g$ is differentiable at $a$ and*

$$D\left(\frac{f}{g}\right)_a = \frac{g(a)\,Df_a - f(a)\,Dg_a}{g(a)^2}$$

**Theorem 21** (Componentwise Nature of Differentiability). *$A \subseteq \mathbb{R}^n, a \in A$ interior point of $A$, $f\colon A \longrightarrow \mathbb{R}^n$, $f\colon A \longrightarrow \mathbb{R}^m$. $f = (f_1, \cdots, f_m)$ component functions of $f$, $f_i\colon A \longrightarrow \mathbb{R}$.*

*$f$ is differentiable at $a \iff$ each $f_i$ for $i \in \{1, 2, \cdots, m\}$ is differentiable at $a$.*

*Proof.* ( $\Longrightarrow$ ) Assume $f$ is differentiable at $a$, so,

$$f(a+h) - f(a) - Df_a(h) \qquad \text{is } o(h)$$

By size bounds,

$$|f_i(a+h) - f_i(a) - (Df_a(h))_i| \leq |f(a+h) - f(a) - Df_a(h)| \qquad = o(h)$$

If a function above is bounded by $o(h)$, then the function also must be $o(h)$. It follows that $f_i(a+h) - f_i(a) - (Df_a(h))_i$ is $o(h)$. Hence, $D(f_i)_a$ exists and

$$D(fi)_a(h) = (Df_a(h))_i$$

( $\Longleftarrow$ ) Suppose $|f_i(a+h) - f_i(a) - (Df_a(h))|$ is $o(h)$ for each $i = 1, \cdots, m$. Then,

$$|f(a+h) - f(a) - Df_a(h) \leq \sum_{i=1}^{m} |f_i(a+h) - f_i(a) - (Df_a(h))_i|$$

Notice that the right hand side is simply sum of $o(h)$, thus, $f(a+h) - f(a) - Df_a(h)$ is $o(h)$. ☐

## 5.4 Partial Derivatives and the Jacobian

The *Jacobian* is the matrix of $Df_a$.

**Definition 21** (Partial Derivative). $A \subseteq \mathbb{R}^n, a \in A$ interior point of $A$. $f\colon A \longrightarrow \mathbb{R}$. For any $j \in \{1, \cdots, n\}$ define $\varphi_j\colon \mathbb{R} \longrightarrow \mathbb{R}, \varphi_j(t) = f(a_1, \cdots, a_{j-1}, t, a_j, \cdots, a_n)$. The $j^{\text{th}}$ partial derivative of $f$ at $a$ is

$$D_j f(a) = \varphi_j'(a)$$

This is under under the assumption that the right side derivative exists. Equivalently, we can write the $j^{\text{th}}$ partial derivative as

$$D_j f(a) = \lim_{t \to 0} \frac{f(a + te_j) - f(a)}{t}, t \in \mathbb{R}$$

Equivalently,

$$f(a + te_j) + D_j f(a) \cdot t + o(t)$$

Computing partial derivatives is relatively easy. Just pretend all other variables are constant.

**Example 20.** $f(x, y, z) = e^y \cos x + z$.

First, pretend $y, z$ are constants, $D_1 f(x, y, z) = -e^y \sin x$. Next, pretend $x, z$ are constants, $D_2 f(x, y, z) = e^y \cos x$. Last, pretend $x, y$ are constants, $D_3 f(x, y, z) = 1$.

Partial derivatives may be more commonly seen in the classical partial derivative notation. See below

$$D_1 f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)$$
$$D_2 f(x,y,z) = \frac{\partial f}{\partial y}(x,y,z)$$
$$D_2 f(x,y,z) = \frac{\partial f}{\partial z}(x,y,z)$$

$x, y, z$ are overloaded here in the classical notation but it's good to be aware of this notation.

## 5.5 Necessary & Sufficient Condition Theorem

**Theorem 22** (Necessary Condition)**.** *$A \subseteq \mathbb{R}^n, a \in A$ interior point of $A, f\colon A \to \mathbb{R}^m$ (vector-valued). If $f$ is differentiable at $a$, then every partial derivative $D_j f_i(a)$ exists $i \leq k \leq n, 1 \leq i \leq m$. Morover $D_j f_i(a)$ is the $(i,j)$ entry of the Jacobian matrix $f'(a)$, so*

$$f'(a) = [D_j f_i(a)]_{m\times n} = \begin{bmatrix} D_1 f_1(a) & D_2 f_1(a) & \cdots & D_n f_i(a) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_m(a) & D_2 f_m(a) & \cdots & D_n f_m(a) \end{bmatrix}$$

*Proof.* We want to show that $D_j f_i(a)$ exists.

$$f'(a)\, e_j = j^{\text{th}} \text{ column of } f'(a) \implies \left(f'(a)\right)_{ij} = \left(f'(a)\, e_j\right)_i$$

Since $f$ is differentiable at $a$, $f(a+h) - f(a) - Df_a(h)$ is $o(h)$. Specialize $h = te_j, f(a+te_j) - f(a) - Df_a(te_j)$ is $o(te_j) - o(t)$. Every coordinate function is also $o(t)$. It follows that

$$\implies f_i(a+te_j) - f_i(a) - t\left(f'(a)\, e_j\right)_i \text{ is } o(t)$$

i.e., $\left(f'(a)\, e_j\right)_i$ satisfies the definition of $D_j f_i(a)$. □

**Remark.** Converse of the theorem is not true: each partial exists at $a \not\rightarrow$ differentiable at $a$.

**Example 21.**

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$D_1 f(0,0), D_2 f(0,0)$ both exist and $= 0$. (on coordinate axes, $f$ is constant $= 0$.) However, the function is not even continuous at $(0,0)$ so it's not differentiable at $(0,0)$. When $Df_a$ exists, we must have $f'(a) = [D_j f_i]_{m\times n}$ i.e., we know its Jacobian matrix. This is the *necessary condition.* What about sufficient condition? That is to say, what is sufficient to show that $Df_a$ exists.

**Theorem 23** (Sufficient Condition)**.** *$A \subseteq \mathbb{R}^n, a \in A$ interior point of $A, f\colon A \longrightarrow \mathbb{R}^m$. Suppose every partial derivative of $f$ exists on some ball around $a$ and each is continuous at $a$. Then, $f$ is differentiable at $a$.*

*Proof.* By componentwise nature of differentiability, we may assume $m = 1$ (so $f$ is real-valued without any loss of generality).

For simplicity of notation, we assume $n = 2$. (This does lose generality but the argument for general cases is very similar).

Interior point $(a,b) \in A \subseteq \mathbb{R}^2$. By the necessary condition theorem, we know that our only candidate for $f'(a,b)$ is

$$\begin{bmatrix} D_1 f(a,b) & D_2 f(a,b) \end{bmatrix}$$

(simplified the case to 2 variables and real valued). I.e., we have

$$Df_{(a,b)}(h,k) = \begin{bmatrix} D_1 f(a,b) & D_2 f(a,b) \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = hD_1 f(a,b) + kD_2 f(a,b)$$

$$\begin{aligned}
f(a+h,b+k) - f(a,b) &= (f(a+h,b+k) - f(a,b+k)) + (f(a,b+k) - f(a,b)) \\
&= (D_1 f(a,b+k) \cdot h + o(h)) + (D_2 f(a,b) \cdot h + o(k)) \\
&= D_1 f(a,b+k) \cdot h + D_2 f(a,b) \cdot k + o(h) + o(k) \\
&= (D_1 f(a,b) + o(1)) \cdot h + D_2 f(a,b) \cdot k + o(h,k) \\
&= D_1 f(a,b) \cdot h + o(1) \cdot h + D_2 f(a,b) \cdot k + o(h,k) \\
&= D_1 f(a,b) \cdot h + o(1) \cdot o(h) + D_2 f(a,b) \cdot k + o(h,k) \\
&= D_1 f(a,b) \cdot h + o(h,k) + D_2 f(a,b) \cdot k + o(h,k) \\
&= D_1 f(a,b) \cdot h + D_2 f(a,b) \cdot k + o(h,k) \\
&= \begin{bmatrix} D_1 f(a,b) & D_2 f(a,b) \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} + o(h,k)
\end{aligned}$$

□

To summarize, the *necessary condition* theorem suggests

$(f$ differentiable at $a) \implies \big($ all partial derivatives exist at $a$ and $f'(a) = [D_i f_i(a)]\big)$

Converse false. The *sufficient condition* theorem suggests

(All partial derivatives at $a$/near $a$ and are continuous at $a$) $\implies$ ($f$ differentiable at $a$)

Converse false. See next example.

**Example 22.**

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

Easy to show that $f$ is differentiable at the origin and $D_f(0,09)$ is the zero transformation.

$$\begin{aligned}
&|f(0+h,0+k) - f(0,0) - 0| = |f(h,k)| \\
&\qquad = |h^2+k^2| |\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right)| \leq 1 (h,k)|^2 \cdot 1 = o(h,k)
\end{aligned}$$

So, $f$ is differentiable at $(0,0)$ but the partials are at continuous at $(0,0)$. We will show for $D_1 f$. $f(x,0) = x^2 \sin\left(\frac{1}{|x|}\right) = x^2 \sin\left(\frac{1}{x}\right)$ when $x > 0$. Here, the partial derivatives are

$$\begin{aligned} D_1 f(x,0) &= 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x}\right) \\ &= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \end{aligned}$$

What you will notice is that $2x \sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0^+$ while $\cos\left(\frac{1}{x}\right)$ oscillates wildly as $x \to 0$. The limit does not exist as $x \to 0^+$.

**Example 23.**

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Away from $(0,0)$,

$$D_1 f(x,y) = \frac{\left(x^2+y^2\right)(2xy) - x^2 y (2x)}{\left(x^2+y^2\right)^2} = \frac{2xy^3}{\left(x^2+y^2\right)^2}$$

$$D_2 f(x,y) = \frac{\left(x^2+y^2\right)x^2 - x^2 y (2y)}{\left(x^2+y^2\right)^2} = \frac{x^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

So, partials exist on $\mathbb{R}^2 - \{(0,0)\}$ and are continuous (rational) functions on $\mathbb{R}^2 - \{(0,0)\}$. Thus, by the sufficient condition theorem, $f$ is differentiable at every $(a,b) \neq (0,0)$, and

$$f'(a,b) = \begin{bmatrix} \frac{2ab^3}{\left(a^2+b^2\right)^2} & \frac{a^2\left(a^2-b^2\right)}{\left(a^2+b^2\right)^2} \end{bmatrix}$$

The case for $(0,0)$: partials are not continuous at $(0,0)$.

$$|D_1 f(h,0) - f(0,0) - 0| = |0-0-0| = 0 = o(h)$$

$$|D_2 f(0,k) - f(0,0) - 0| = |0-0-0| = 0 = o(k)$$

Thus, $D_1 f(0,0) = D_2 f(0,0) = 0$. So, if $f$ is differentiable at 0, we must have

$$f'(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e., $Df_{(0,0)}$ is the zero transformation. Looking at the line $h = k$, then,

$$\left|\frac{h^3}{h^2+h^2}\right| = \frac{|h|^3}{2|h|^2} = \frac{|h|}{2}$$

This is not $o(h)$ since $\frac{|h|}{2|h|} \to \frac{1}{2} \neq 0$. Thus, $f$ is not differentiable at $(0,0)$. It follows that the only $o(h)$ is possible if $Df_{(0,0)}$ is the zero transformation, however, we just showed that it is not $o(h)$.

By Chain Rule, we know

$$D\left(g\circ f\right)_a = Dg_f\left(a\right)\circ Df_a$$

We want to now know the entries of the Jacobian matrix of $\left(g\circ f\right)'\left(a\right)$. We know from the necessary condition theorem that the $(i,j)$ entry of $\left(g\circ f\right)'\left(a\right)$ is $D_j\left(g\circ f\right)_i\left(a\right)$.

**Theorem 24.** *$A\subseteq\mathbb{R}^n, a\in A$ interior point of $A, f\colon A\longrightarrow\mathbb{R}^m, f\left(a\right)\subseteq\mathcal{B}\subseteq\mathbb{R}^m, g\colon\mathcal{B}\longrightarrow\mathbb{R}^l, f\left(\right)\in\mathcal{B}$ interior of $\mathcal{B}$. If $f$ is differentiable at $a$ and $g$ is differentiable at $f\left(a\right)$, then, $g\circ f$ is differentiable at $a$, so all partials exist and*

$$D_j\left(g\circ f\right)_i = \sum D_k g_j\left(f\left(a\right)\right)\cdot D_j f_k\left(a\right)$$

*Proof.* From regular chain rule in single variable,

$$\begin{aligned}\left(g\circ f\right)'\left(a\right) &= \left[D_j\left(g\circ f\right)_i\right]_{l\times n} = g'\left(f\left(a\right)\right)f'\left(a\right)\\ &= \left[D_k g_i\left(f\left(a\right)\right)\right]_{l\times m}\left[D_j f_k\left(a\right)\right]_{m\times n}\\ &= \left[D_k g_i\left(f\left(a\right)\right)\right]_{l\times m}\left[D_j f_k\left(a\right)\right]_{m\times n}\\ &= \left[\sum_{k=1}^{m} D_k g_i\left(f\left(a\right)\right)D_j f_k\left(a\right)\right]_{l\times n}\end{aligned}$$

Notice that the above and $\left[D_j\left(g\circ f\right)_i\left(a\right)\right]_{l\times n}$ are equal entry by entry. ☐

In terms of classical notation, for $x = x\left(s,t\right), y = y\left(s,t\right), z = z\left(s,t\right)$, we have,

$$f\left(x,y,z\right) = f\left(x\left(s,t\right),y\left(s,t\right),z\left(s,t\right)\right)$$

Then,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\times\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\times\frac{\partial y}{\partial s}+\frac{\partial f}{\partial z}\times\frac{\partial z}{\partial s}$$

Do note that we are not dividing quantities, rather, we're applying an operator $\frac{\partial}{\partial x}$ to $f$ s.t. $\frac{\partial}{\partial x}\left(f\right) = \frac{\partial f}{\partial x}$.

## 5.6 Differential Curves

A function of the form $\gamma = \mathbb{R}\longrightarrow\mathbb{R}^m$ is called a curve in $\mathbb{R}^m$. The domain of $\gamma$ could also be the same interval in $\mathbb{R}$. As such, $D\gamma_t\colon\mathbb{R}\longrightarrow\mathbb{R}^m$, so, $\gamma'\left(t\right)$ is a $m\times 1$ matrix, i.e., a column vector. Thus,

$$\gamma'\left(t\right)\in\mathbb{R}^m, \gamma = \left(\gamma_1,\cdots,\gamma_m\right), \gamma_i\colon\mathbb{R}\longrightarrow\mathbb{R}$$

and

$$\gamma'\left(t\right) = \begin{bmatrix}\gamma_1'\left(t\right)\\ \gamma_2'\left(t\right)\\ \vdots\\ \gamma_m'\left(t\right)\end{bmatrix}$$

We also have,

$$\gamma\left(t+h\right) = \gamma\left(t\right)+\gamma'\left(t\right)h+o\left(h\right)$$

In $\mathbb{R}^2,\mathbb{R}^3$, especially, $\gamma'\left(t\right)$ is called the velocity vector at time $t$, $\left|\gamma'\left(t\right)\right|$ is called the speed.

**Example 24.** $\gamma\colon\mathbb{R}\longrightarrow\mathbb{R}^2, \gamma\left(t\right) = \left(\cos t,\sin t\right)$. $\gamma$ is called the parametrization of $S^1$ = unit circle in $\mathbb{R}^2$.

$$\gamma'\left(t\right) = \begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}$$

Note $\langle \gamma(t), \gamma'(t) \rangle = 0$, always orthogonal. Thus,

$$|\gamma'(t)| = 1$$

$\gamma$ is at unit speed.

Let's see antoher parametrization of $S$,

$$\beta = \mathbb{R} \longrightarrow \mathbb{R}^2, \beta(t) = \gamma(2t) = (\cos(2t), \sin(2t))$$

$$\beta'(t) = \begin{bmatrix} -2\sin(2t) \\ 2\cos(2t) \end{bmatrix}, |\beta'(t)| = 2 \forall t$$

**Proposition.** Suppose $\gamma \colon \mathbb{R} \longrightarrow \mathbb{R}^m$ differentiable curve such that $\langle \gamma(t), \gamma'(t) \rangle = 0 \forall t \in \mathbb{R}$. Then $|\gamma(t)|$ is constant (as a function of $t$).

*Proof.* Set $f(t) = |\gamma(t)|^2$. Since $|\gamma(t)| \geq 0 \forall t$, it is enough to show that $f(t)$ is constant.

$$|\gamma(t)|^2 = \langle \gamma(t), \gamma'(t) \rangle$$

$$(\gamma, \gamma) : \mathbb{R} \longrightarrow \mathbb{R}^m, (\gamma, \gamma)(t) = (\gamma(t), \gamma(t))$$

Let $g \colon \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be $g(x, y) = \langle x, y \rangle$. Thus, $f = g \circ (\gamma, \gamma)$. By the chain rule,

$$Df_t = \underbrace{Dg_{(\gamma(t),\gamma(t))}}_{Dg_{(\gamma(t),\gamma(t))}(h,k) = \langle h, \gamma(t) \rangle + \langle \gamma(t), k \rangle} \circ \overbrace{D(\gamma, \gamma)_t}^{D(\gamma(t),\gamma(t))_t h = (\gamma'(t), \gamma'(t))_h}$$

$$Df_t(h) = Dg_{(\gamma(t),\gamma(t))}\left(h\left(\gamma'(t), \gamma'(t)\right)\right) = h\left(2\langle \gamma(t), \gamma'(t) \rangle\right) = 0$$

$Df_t$ is zero transformation $\implies f'(t) = 0 \forall t$. $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ has zero derivative everywhere. Hence, $f$ is a constant function. $\square$

### 5.6.1 Parametrization of Lines

Let $L$ be a line in $\mathbb{R}^n$. If $p, q \in L$, then $v = q - p$ is a vector parallel to $L$ (i.e., $v$ is the direction of $L$).

$$\begin{aligned} \gamma \colon \mathbb{R} \longrightarrow \mathbb{R}^n, \gamma &= p + tv \\ &= p + t(q - p) \\ &= (1 - t)p + tq \end{aligned}$$

Note that if we restrict to $\gamma$ to have domain $[0, 1]$, then $\gamma \colon [0, 1] \to \mathbb{R}^n, \gamma(t) = (1 - t)p + tq$, then this is a parametrization of the line segment starting at $p$ and ending at $q$.

## 5.7 Higher-Order Derivatives

$$\begin{aligned} f\mathbb{R}^n \longrightarrow \mathbb{R},\ & D_1 f \colon \mathbb{R}^n \longrightarrow \mathbb{R} && \text{assuming partial exists} \\ & D_2(D_1 f) : \mathbb{R}^n \longrightarrow \mathbb{R} && \text{second order partial} \\ & D_1(D_2(D_1 f)) : \mathbb{R}^n \longrightarrow \mathbb{R} && \text{third order partial} \end{aligned}$$

The point is that we can iterate partials when they exist.

**Notation.**

$$D_i(D_k f) = D_i D_k f = D_{ki} f$$

The notation suggests to first do $k$, then do $i$.

Let $w = f(x, y, z)$, then, all the following mean the same thing,

$$f_{233}, f_{yzz}, \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\right)\right), \frac{\partial^3}{\partial z^2 \partial y} f, w_{yzz}, \frac{\partial^3}{\partial z^2 \partial y} w$$

**Definition 22.** $f$ is $\mathcal{C}^k$ if all fo its $k^{\text{th}}$ order partial derivatives exist and are continuous at all points of domain of $g$.

**Remark.** $f$ is $\mathcal{C}^\infty$ (or *smooth*) if partial derivatives of all orders exist (and are continuous).

**Theorem 25.** *$A \subseteq \mathbb{R}^2, f\colon A \longrightarrow \mathbb{R}$ is $\mathcal{C}^2$. Then, at every interior point $a \in A$*

$$D_{12} f(a, b) = D_{21} f(a, b)$$

*Proof. (Sketch).* Since $(a, b) \in A$, there are $h, k$ small s.t.

$$C = [a, a+h] \times [b, b+k] \subseteq A$$

It's clear that $C$ is compact.

$$\begin{aligned}
\int_a^{a+h} \int_b^{b+k} D_{12} f(x, y)\; dy\, dx &= \int_a^{a+h} \int_b^{b+k} D_2 \left(D_1 f(x, y)\right)\, dy\, dx \\
&\overbrace{=}^{FTC} \int_a^{a+h} \left(D_1 f(x, b+x) - D_1 f(x, y)\right)\, dx \\
&\overbrace{=}^{FTC} f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a+b) =: \Delta(h, k)
\end{aligned}$$

By Extreme Value Theorem, the following exist on $C$:

$$m_{h,k} = \min\left\{D_{12} f(x, y) :\ (x, y) \in C\right\}$$

or

$$C\colon m_{h,k} \leq D_{12} f(x, y) \leq M_{h,k}$$

where

$$M_{h,k} = \max\left\{D_{12} f(x, y) :\ (x, y) \in C\right\}$$

If we integrate all parts:

$$\int_a^{a+h} \int_b^{b+k} m_{h,k}\, dy\, dx \leq \int_a^{a+h} \int_b^{b+k} D_{12} f(x, y)\; dy\, dx \leq \int_a^{a+h} \int_b^{b+k} M_{h,k}\, dy\, dx$$

$$m_{h,k}(hk) \leq \Delta(h, k) \leq M_{h,k}(hk)$$

$$\Longrightarrow\ m_{h,k}(hk) \leq \frac{\Delta(h, k)}{hk} \leq M_{h,k}(hk)$$

Let $(h, k) \longrightarrow 0$. By continuity,

$$\begin{aligned}
M_{h,k} &\longrightarrow D_{12} f(a, b) \\
m_{h,k} &\longrightarrow D_{21} f(a, b)
\end{aligned}$$

Then, by squeeze,

$$\frac{D(h, k)}{hk} \longrightarrow \Delta_{12} f(a, b)$$

But, we can repeat the same argument with variables reversed to get $\frac{\Delta(h,k)}{hk} \longrightarrow$

$D_{21}f(a,b)$ as $(h,k) \longrightarrow (0,0)$. By uniqueness of limits

$$D_{12}f(a,b) = D_{21}f(a,b)$$

$\square$

**Remark.** For small enough $h, k$, as the square gets smaller and smaller,

$$\int_a^{a+h}\int_b^{b+k} g(x,y)\, dy\, dx \approx g(a,b)\cdot hk$$
$$\int_b^{b+k}\int_a^{a+h} w(x,y)\, dy\, dx \approx w(a,b)\cdot hk$$

## 5.8 Extreme Points for Multivariable Functions

Our goal in this section is to generalize the second derivative test.

Let $A \subseteq \mathbb{R}, f\colon A \longrightarrow \mathbb{R}, f$ is $\mathcal{C}^2$ (twice continuously differentiable) on the interior point of $A$. If $a \in A$ is an interior point of $A$ and $a$ is a critical point of $f$, (i.e., $f'(a) = 0$), then:

(1) If $f''(a) > 0$, then $f(a)$ is a *local minimum* for $f$.
(2) If $f''(a) < 0$, then $f(a)$ is a *local maximum* for $f$.
(3) If $f''(a) = 0$, the second derivative test is *inconclusive*.

### 5.8.1 The Hessian Matrix: Second Order Matrices

$A \subseteq \mathbb{R}^n, f\colon A \longrightarrow \mathbb{R}, a \in A$, $f$ is $\mathcal{C}^2$ on its interior points. $Df_a\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a linear transformation and $f'(a)$ is a $1\times n$ matrix where

$$f'(a) = \begin{bmatrix} D_1f(a) & \cdots & D_nf(a)\end{bmatrix}$$

We think about $f'$ as a function from its interior points to $\mathbb{R}^n$ with component functions $D_1f, D_2f, \cdots, D_nf$. Then, $f''(a)$ is a $n\times n$ square matrix,

$$\begin{aligned} f''(a) &= \begin{bmatrix} D_1(D_1f)_a & D_2(D_1f)_a & \cdots & D_n(D_1f)_a \\ D_1(D_2f)_a & D_2(D_2f)_a & \cdots & D_n(D_1f)_a \\ \vdots & \vdots & \ddots & \vdots \\ D_1(D_nf)_a & D_2(D_nf)_a & \cdots & D_n(D_nf)_a \end{bmatrix} \\ &= \begin{bmatrix} D_{11}f(a) & D_{12}f(a) & \cdots & D_{1n}f(a) \\ D_{21}f(a) & D_{22}f(a) & \cdots & D_{2n}f(a) \\ \vdots & \vdots & \ddots & \vdots \\ D_{n1}f(a) & D_{n2}f(a) & \cdots & D_{nn}f(a) \end{bmatrix} \end{aligned}$$

Since $f$ is $\mathcal{C}^2$, equality of mixed partials, so this matrix is symmetric (across its diagonal).

**Definition 23** (Symmetric)**.** A (necessarily) square $n\times n$ matrix $A$ is *symmetric* if $A_{ij} = A_{ji}\ \forall i \leq i, j \leq n$.

We can also express symmetric matrices using *transposes*.

**Definition 24** (Transpose)**.** Let $M$ be $m\times n$ matrix. The *transpose* of $M$ is the $n\times m$ matrix, denoted $M^T$, defined

$$\left(M^T\right)_{ij} = M_{ji}$$

for $1 \leq i \leq n, 1 \leq j \leq m$.

I.e., the 1st row of $M$ is the 1st column of $M^T$, the 2nd row of $M$ is the 2nd column of $M^T$, etc. Note that a matrix $A$ is symmetric iff $A^T = A$.

**Example 25.** (1) The matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, $A^T = A$, so $A$ is symmetric.

(2) The matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

(3) The matrix $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $x^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

In general, if $x$ is a column vector, $x^T$ is a row vector. For $x, y \in \mathbb{R}^n$ column vectors,

$$\begin{aligned} x^T y &= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\ &= x_1 y_1 + \cdots + x_n y_n \\ &= \langle x, y \rangle \end{aligned}$$

**Theorem 26.** *$A \subseteq \mathbb{R}^2, f\colon A \longrightarrow \mathbb{R}$, a $\mathcal{C}^2$ function on interior points of $A$. $(a,b) \in A$ interior point and $f'(a,b) = \begin{bmatrix} 0 & 0 \end{bmatrix}$ (i.e. $(a,b)$ is a critical point for $f$.)*

*Let $f''(a,b) = \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix}$, then,*

*(a) If $\alpha > 0$ and $\alpha\delta - \beta^2 > 0$, then $f(a,b)$ is a local minimum.*

*(b) If $\alpha < 0$ and $\alpha\delta - \beta^2 > 0$, then $f(a,b)$ is a local maximum.*

*(c) If $\alpha\delta - \beta^2 < 0$, then $f(a,b)$ is a saddle point.*

*(d) If $\alpha\delta - \beta^2 = 0$, then no conclusion.*

**Example 26.** $f(x,y) = \sin^2 x + x^2 y + y^2$

We have $f_x(x,y) = 2\sin x + 2xy$ and $f_y(x,y) = x^2 + 2y$. We must have critical points satisfying

$$\begin{cases} 2\sin x + 2xy = 0 \\ x^2 + 2y = 0 \end{cases}$$

Note that this is not a system of linear equations. In general, it's hard to find all critical points but $(0,0)$ is one critical point so we classify it.

$$\begin{aligned} f''(x) &= \begin{bmatrix} 2\cos^2 x - 2\sin x + 2y & 2x \\ 2x & 2 \end{bmatrix} \\ f''(0,0) &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

We note that $\alpha = 2 > 0, \alpha\delta - \beta^2 = 4 > 0$, so $(0,0)$ is a local minimum.

**Example 27.** $f(x,y) = xy(x+y-3)$ . We want to optimize $f$ on

$$T\colon \{(x,y) : x \geq 0, y \geq 0, x + y \leq 3\}$$

We begin by noting that $f$ has maximum and minimum on $T$ by extreme value theorem since $f$ is continuous and $T$ is compact. Also note that $f = 0$ on all 3 boundaries of $T$;

in interiors of $T$, $x > 0, y > 0, x + y < 3 \implies x + y - 3 < 0$. So $f(x, y) < 0$ whenever $(x, y)$ is in interior of $T$.

Thus, max value of $f$ on $T$ is 0 and it is achieved on boundary of $T$. So, we need to find the minimum by using the new theorem (just on interior points).

$$f_x(x, y) = y(2x + y - 3) = 0 \text{ and } f_y(x, y) = x(2y + x - 3) = 0 \iff \begin{cases} 2x + y - 3 = 0 \\ x + 2y - 3 = 0 \end{cases}$$

since $x > 0, y > 0$ in interior of $T$. The above is true $iff x = 1, y = 1$. So $(1, 1)$ is the only critical point in interior of $T$.

$$f''(x, y) = \begin{bmatrix} 2x & 2x + 2y - 3 \\ 2x + 2y - 3 & 2x \end{bmatrix}, f''(1, 1) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$\alpha = 2 > 0, \alpha\delta - \beta^2 = 3 > 0$, hence $f(1, 1)$ is minimum on interior of $T$.

**Theorem 27** (Critical Point Theorem)**.** *$A \subseteq \mathbb{R}^2, a \in A$ interior point, $f\colon A \longrightarrow \mathbb{R}$. If $f$ is differentiable at $a$ and $f$ has a local min/max at $a$, then $a$ is a critical point for $f$, i.e., $f''(a) =$ zero matrix.*

*Proof. (Local Min/Max Theorem in 2 Variables).* It is enough to show $D_j f(a) = 0$ for $j \in \{1, 2, \cdots, n\}$. Fix $j \in \{1, 2, \cdots, n\}$. Recall $\varphi_j(x) = f(a_1, a_2, \cdots, a_j, x, a_{j+1}, \cdots, a_n)$.

Since $a$ is a local min/max for $f$, it follows that $\varphi_j$ has a local min/max at $a_j$. By single variable calculus, $D_j f(a) = \varphi_j'(a_j) = 0$.

Proof of local mon/max theorem requires a generalization of taylor's theorem. □

### 5.8.2 Quadratic Forms

**Definition 25** (Quadratic Form)**.** Let $A$ be $n \times n$ square matrix. The *quadratic form* induced by $A$ is $\mathcal{Q}_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}, \mathcal{Q}_A(h) = \langle h, Ah \rangle = h^T Ah$.

Note that $\mathcal{Q}_A(th) = \langle th, Ath \rangle = t^2 \langle h, Ah \rangle = t^2 \mathcal{Q}_a(h)$. The most important fact for us if $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, \mathcal{Q}_{f''(a)}$, which we will denote by $\mathcal{Q}f_a, \mathcal{Q}f_a(h) = h^T f''(a) h$ (assuming $f''(a)$ exists).

**Theorem 28** (Quadratic Taylor Approximation)**.** *$I \subseteq \mathbb{R}$, open interval, $[0, 1] \subseteq I, \varphi\colon I \longrightarrow \mathbb{R}, \mathcal{C}^2$ function, then,*

$$\varphi(1) = \varphi(0) + \varphi'(0) + \frac{1}{2}\varphi''(c)$$

*for some $c \in [0, 1]$.*

We will assume this and use it to prove a multivariable version.

**Theorem 29.** *$A \subseteq \mathbb{R}^n, f\colon A \longrightarrow \mathbb{R}, \mathcal{C}^2$ function on interior points of $A$. $a \in A$ interior point, then, for small enough $h$,*

$$f(a + ch) = f(a) + Df_a(h) + \frac{1}{2}\mathcal{Q}f_{a+ch}(h)$$

*for some $c \in [0, 1]$.*

The theorem essentially shows that the expression is just an $o(h)$ term but it's a nice way to represent $o(h)$ in terms of a quadratic form.

*Proof.* Fix small enough $h$ so that $A$ contains all the points on line segment from $a$ to $a+h$. Define $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}^n, \gamma(t) = (1-t)a + t(a+h) = a+th, \gamma(0) = a, \gamma(1) = a+h$.

Define $\gamma\colon \mathbb{R} \longrightarrow \mathbb{R}, \gamma(t) = f(\gamma(t))$. By quadratic taylor in single variable, there is $c \in [0,1]$ s.t.

$$\varphi(1) = \varphi(0) + \varphi'(0) = \frac{1}{2}\varphi''(c)$$

Since

$$\begin{aligned} f(\gamma(1)) &= f(a+ch) \\ f(\gamma(0)) &= f(a) \\ \varphi'(0) = f'(\gamma(0))h &= f'(a)h = Df_a h \end{aligned}$$

We are just left to show that $\varphi''(c) = \mathcal{Q}f_{a+ch}(h)$. Using chain rule and the derivative of the inner product,

$$\begin{aligned} \varphi''(t) &= \langle (f'(\gamma(t)))', h\rangle + \langle f'(\gamma(t)), 0\rangle \\ &= \langle f''(\gamma(t))\gamma'(t), h\rangle = \langle h, f''(\gamma(t))h\rangle = \mathcal{Q}f_{\gamma(t)}(h) \end{aligned}$$

And so,

$$\varphi''(c) = \mathcal{Q}f_{a+ch}(h)$$

□

**Remark.** $f(a+h) = f(a) + Df_a(h) + \frac{1}{2}\mathcal{Q}f_{a+ch}(h)$, a critical point, then $Df_a(h) = 0$. If $a$ critical point point at $\mathcal{Q}f_{a+ch}(h) \geq 0 \forall h$, then $f(a+h) = f(a) + \frac{1}{2}\mathcal{Q}f_{a+ch}(h) \geq f(a) \implies a$ is local minimum.

**Definition 26.** Let $M$ be $n \times n$ symmetric matrix,

(1) $M$ is *positive definit* if $\mathcal{Q}_M(h) > 0$ for every non-zero $h$.
(2) $M$ is *negative definite* if $\mathcal{Q}_M(h) < 0$ for every non-zero $h$.
(3) Otherwise, $M$ is *indefinite* if $\mathcal{Q}_m(h) \geq 0$ for some $h$, $\mathcal{Q}_M(h) < 0$ for some $h$.

**Example 28.** $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathcal{Q}_M(h,k) = \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = h^2 + k^2$. $M$ is positive definite.

**Example 29.** $M = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}, \mathcal{Q}_M(e_1) = e_1^T \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} e_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = -2 < 0, \mathcal{Q}_M(e_2) > 0$. $M$ is indefinite.

**Proposition.** Let $M = \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix}$,

(1) $M$ is *positive definite* $\iff \alpha > 0, \alpha\delta - \beta^2 > 0$.
(2) $M$ is *negative definite* $\iff \alpha < 0, \alpha\delta - \beta^2 > 0$.
(3) $M$ is *indefinite* $\iff \alpha\delta - \beta^2 < 0$.

# 6 Directional Derivatives and the Gradient

$A \subseteq \mathbb{R}^n, a \in A$ interior point of $A$, $f\colon A \longrightarrow \mathbb{R}$. Let $j \in \{1, \cdots, n\}$. One definition of the $j^{\text{th}}$ partial derivative of $f$ at $a$ is

$$D_j f(a) = \lim_{t \to 0} \frac{f(a+te_j) - f(a)}{t}$$

$D_j f(a)$ signifies the rate of change of $f$ at $a$ when the input varies in the $e_j$ direction. We want to generalize this defintiion to any direction, not just in the general coordinate axes direction.

By direction, we mean a unit vector $d \in \mathbb{R}^n$, that is,

$$|d| = 1$$

**Definition 27** (Directional Derivative)**.** $A \subseteq \mathbb{R}^n, a \in A$, interior point, $f\colon A \longrightarrow \mathbb{R}, d \in \mathbb{R}^n$ a unit vector. The *directional derivative* of $f$ at $a$ in the direction $d$ is

$$D_d f(a) = \lim_{t \to 0} \frac{f(a+td) - f(a)}{t}$$

assuming this limit exists.

**Remark.** When $d$ is a standard basis vector, i.e. $d = e_j$, we use notation

$$D_j f(a) = De_j f(a)$$

and call it the $j^{\text{th}}$ partial derivative of $f$ at $a$.

Assume now that $f\colon A \longrightarrow \mathbb{R}$ is differentiable at $a$. A special case of the differentiability definition gives us

$$f(a+h) - f(a) - Df_a(h) \qquad \text{is } o(h)$$

Taking $h = td$,

$$f(a+td) - f(a) - Df_a(td) \qquad \text{is } o(td) = o(t)$$

Since $|td| = |t||d| = |t|$. It follows that,

$$\lim_{t \to 0} \frac{f(a+td) - f(a) - tDf_a(d)}{t} = 0$$

$$\lim_{t \to 0} \frac{f(a+td) - f(a)}{t} = Df_a(d) \implies D_d f(a) = Df_a(d)$$

$$\begin{aligned}
\implies D_d f(a) = f'(a)\, d &= \begin{bmatrix} D_1 f(a) & \cdots & D_n f(a) \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \\
&= \sum_{j=1}^{n} D_j f(a)\, d_j \\
&= \langle \begin{bmatrix} D_1 f(a) \\ \vdots \\ D_n f(a) \end{bmatrix}, \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \rangle
\end{aligned}$$

**Definition 28** (Gradient)**.** The *gradient* of $f\colon A \longrightarrow \mathbb{R}, A \subseteq \mathbb{R}^n$ at $a$ is

$$\nabla f(a) = f'(a)^T$$

It follows that,

$$\begin{aligned}
D_d f(a) = Df_a(d) = f'(a)\, d &= \begin{bmatrix} D_1 f(a) & \cdots & D_n f(a) \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \\
&= \sum_{j=1}^{n} D_j f(a)\, d_j \\
&= \langle \begin{bmatrix} D_1 f(a) \\ \vdots \\ D_n f(a) \end{bmatrix}, \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \rangle \\
&= \langle \nabla f(a), d \rangle
\end{aligned}$$

Still assuming $f\colon A \longrightarrow \mathbb{R}$ is differentiable at $a$, what are the possible values of $D_d f(a)$? (real-valued).

By the Cauchy-Schwarz Inequality,

$$\begin{aligned}
|D_d f(a)| = |\langle \nabla f(a), d \rangle| &\leq |\nabla f(a)| \cdot |d| = |\nabla f(a)| \\
\implies -|\nabla f(a)| &\leq D_d f(a) \leq |\nabla f(a)|
\end{aligned}$$

In fact, $D_d f(a) = |\nabla f(a)|$ when $d$ is parallel to $\nabla f(a)$.

When $\nabla f(a) \neq 0$, set $d = \frac{\nabla f(a)}{|\nabla f(a)|}$ (normalized gradient).

$$\begin{aligned}
D_d f(a) = \langle \nabla f(a), d \rangle &= \langle \nabla f(a), \frac{1}{|\nabla f(a)|} \nabla f(a) \rangle \\
&= \frac{1}{|\nabla f(a)|} \langle \nabla f(a), \nabla f(a) \rangle \\
&= \frac{|\nabla f(a)|^2}{|\nabla f(a)|} = |\nabla f(a)|
\end{aligned}$$

Therefore, large rate of change is achieved when travel in the direction of $\nabla f(a)$. Similar calculation shows that $D_d f(a) = -|\nabla f(a)|$ when

$$d = -\frac{1}{|\nabla f(a)|} \nabla f(a)$$

**Theorem 30.** *$A \subseteq \mathbb{R}^n, a \in A$ interior point of $A, f\colon A \longrightarrow \mathbb{R}, f$ is differentiable at $a$. Then, for every unit vector $d \in \mathbb{R}^n$, directional derivative $D_d f(a)$ exists and*

$$D_d f(a) = f'(a)\, d = \langle \nabla f(a), d \rangle = |\nabla f(a)| \cos \theta_{\nabla f(a), d}$$

Moreover,

$$-|\nabla f(a)| \leq D_d f(a) \leq |\nabla f(a)|$$

with

$$D_d f(a) = |\nabla f(a)|$$

when $d = \frac{\nabla f(a)}{|\nabla f(a)|}$ and

$$D_d f(a) = -|\nabla f(a)|$$

when $d = \frac{-\nabla f(a)}{|\nabla f(a)|}$ whenever $\nabla f(a) \neq 0$.

Furthermore, $\nabla f(a)$ points in the direction of greatest increase ($|\nabla f(a)|$ is the rate of increase) for $f$ at $a$, $-\nabla f(a)$ points in direction of greatest decrease for $fr$ at $a$ with rate $-|\nabla f(a)|$.

In addition, directions orthogonal to $\nabla f(a)$ are directions with rate of change 0 for $f$.

**Remark.** Converse of this theorem is false.

In fact, there is a real value $f$ such that $D_d f(a)$ all exist with $D_d f(a) = \langle \nabla f(a), d\rangle$ for all unit vectors $d$ but $f$ is not differentiable at $a$.

**Example 30.** $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$; $f(x,y) = 100 - x^2 - y^2$.

We start at $(0,1)$, direction to ascend quickest

$$\nabla f(x,y) = \begin{bmatrix} D_1 f(x,y) \\ D_2 f(x,y) \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \end{bmatrix}$$

$$\nabla f(0,1) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, d = \frac{1}{|\nabla f(0,1)|} = \frac{1}{2}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Also note that if you travel in directions orthogonal to $\nabla f(0,1)$ (i.e., in $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$), then, no immediate change in elevation will be noticed.

## 6.1 Level Sets

**Definition 29** (Level Set)**.** Let $A \subseteq \mathbb{R}^n, f\colon A \longrightarrow \mathbb{R}^m, b \in \mathbb{R}^m$. The *level set* of $f$ at $b$ is

$$L_b^f = L_b = \{x \in A\colon f(x) = b\}$$

Recall that $f\colon A \longrightarrow \mathbb{R}, A \subseteq \mathbb{R}^n, \nabla f(a) = f'(a)^T$.

**Example 31.** $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}, f(x,y,z) = 2x^2 + y^2 + 3z^2$.

graph $(f)$ is a subset of $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ (so we can't draw). What do the level sets look like? (Subsets of $\mathbb{R}^3$).

$$b \in \mathbb{R}, L_b^f = \left\{(x,y,z) : 2x^2 + y^2 + 3z^2 = b\right\}$$

For a real-valued $f$ differentiable at $a$, $\nabla f(a)$ is always orthogonal to $L_{f(a)}^f$.

**Theorem 31.** *Let $A \subseteq \mathbb{R}^n, a \in A$ interior point of $A$, $f\colon A \longrightarrow \mathbb{R}$ differentiable at $a$. $\epsilon > 0, \gamma\colon (-\varepsilon, \varepsilon) \longrightarrow L_{f(a)}, \gamma$ differentiable at $a$ and $\gamma(0) = a$. Then, $\gamma'(0)$ is orthogonal to $\nabla f(a)$.*

*Proof.* Since $\gamma(t) \in L_{f(a)} \forall t \in (-\varepsilon, \varepsilon), f \circ \gamma\colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ is a constant function $\big(f(\gamma(t)) = f(a) \text{ since } \gamma(t) \in L_{f(a)}\big)$

Moreover, by chain rule, $f \circ \gamma$ is differentiable at 0 and

$$\begin{aligned} 0 = (f \circ \gamma)'(0) &= f'(\gamma(0))\,\gamma'(0) = f'(a)\,\gamma'(0) \\ &= \nabla f(a)^T \gamma'(0) = \langle \nabla f(a), \gamma'(0)\rangle \end{aligned}$$

□

The graph of a function $f\colon A \longrightarrow \mathbb{R}^m$ $(A \subseteq \mathbb{R}^n)$ can always be thought of as a level set of some other function. The idea is that

$$(x,y) \in \text{graph}(f) \iff f(x) = y \iff f(x) - y = 0$$

So the graph $(f)$ is just $L_0^{f(x)-y}$.

Define $g\colon A\times\mathbb{R}^m \longrightarrow \mathbb{R}^m$ by $g(x,y) = f(x) - y$. Then $\text{graph}(f) = L_0^g$.

## 6.2 Tangent Hyperplanes

$A \subseteq \mathbb{R}^n, a \in A$ interior point of $A, f\colon A \longrightarrow \mathbb{R}, f$ is differentiable at $a$. We want an equation that describes the hyperplane tangent to graph$(f)$ at $(a, f(a))$.

Define $g\colon A\times\mathbb{R} \longrightarrow \mathbb{R}$, $g(x,y) = f(x) - y$ so $\text{graph}(f) = L_0^g$. Thus, $\nabla g(a, f(a))$ is normal to $L_0^g = \text{graph}(f)$ at $(a, f(a))$. Since $x \in \mathbb{R}^n, y \in \mathbb{R}$,

$$\nabla g(x,y) = \begin{bmatrix} D_1 g(x,y) \\ \vdots \\ D_n g(x,y) \\ D_{n+1} g(x,y) \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \\ -1 \end{bmatrix} = \begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix}$$

$$\nabla g(a, f(a)) = \begin{bmatrix} \nabla f(a) \\ -1 \end{bmatrix}$$

The vectors on the plane tangent to graph$(g)$ at $(a, f(a))$ are exactly the vectors so that when they are translated by $-(a, f(a))$, they become orthogonal to

$$\begin{bmatrix} \nabla f(a) \\ -1 \end{bmatrix}$$

i.e., the vectors $(x,y)$ which satisfy

$$\begin{aligned} &\langle (x,y) - (a, f(a)), \begin{bmatrix} \nabla f(a) \\ -1 \end{bmatrix} \rangle = 0 \\ \iff\ & D_1 f(a)(x_1 - a_1) + \cdots + D_n f(a)(x_n - a_n) - (y - f(a)) = 0 \\ \iff\ & y = f(a) + D_1 f(a)(x_1 - a_1) + \cdots + D_n f(a)(x_n - a_n) \end{aligned}$$

**Example 32.** $f(x,y) = e^x - y$. Find tangent plane for $f$ at $(0, 1, f(0,1))$.

$$D_1 f(x,y) = e^x, D_2 f(x,y) = -1$$

So,

$$\nabla f(0,1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} z &= f(0,1) + D_1 f(0,1)(x - 0) + D_2 f(0,1)(y - 1) \\ &= x - (y-1) \iff x - y - z = -1 \end{aligned}$$

# 7 Inverse & Implicit Functions

## 7.1 The Inverse Function Theorem

**Definition 30** (Open Ball)**.** A subset $A$ is open (where $A \subseteq \mathbb{R}^n$) if every $a \in A$ is an interior point of $A$, i.e., for every $a \in A$, there is $\varepsilon > 0$ s.t. $\mathcal{B}(a, \varepsilon) \subseteq A$.

**Definition 31** (Inverse Function Theorem)**.** Let $A \subseteq \mathbb{R}^n$ be open, let $f\colon A \longrightarrow \mathbb{R}^n$ be $\mathcal{C}^1$, let $a \in A$, and assume $f'(a)$ is an invertible matrix.

Then, there exists an open set $V \subseteq A$ with $a \in V$ and an open set $W \subseteq \mathbb{R}^n$ with

$f(a) \in W$ such that $f \colon V \longrightarrow W$ has a $\mathcal{C}^1$ inverse $f^{-1} \colon W \longrightarrow V$. Moreover, for each $x \in V$, $y = f(x) \in W$, we have

$$D\left(f^{-1}\right)_y = (Df_x)^{-1}, \text{ i.e., } \left(f^{-1}\right)'(y) = \left(f'(x)\right)^{-1}$$

**Remark.** An *affine function* is a function $g$ of the form $g(x) = c + T(x)$ where $c$ is a vector and $T$ is linear.

**Theorem 32.** *Given $f$, $a$ as in the inverse function theorem, $b = f(a)$, then,*

$$\begin{aligned} f^{-1}(b+h) &= f^{-1}(b) + D\left(f^{-1}\right)_b(h) \\ &= a + (Df_a)^{-1}(h) \end{aligned}$$

*is the best affine approximation to $f^{-1}$ near $b = f(a)$.*

**Remark.** We will not prove the Inverse Function Theorem. But the last claim follows easily from what's before it.

$$f^{-1} \circ f \colon V \longrightarrow V, f^{-1} \circ f = \mathrm{id}_v \implies D\left(f^{-1} \circ f\right)_a = D(\mathrm{id}_v)_x = \mathrm{id}_{\mathbb{R}^n}$$
$$\mathrm{id}_{\mathbb{R}^n} = D\left(f^{-1} \circ f\right)_x = D\left(f^{-1}\right)_{f(x)} = Df_x$$

Similarly,

$$\mathrm{id}_{\mathbb{R}_n} = Df_x \circ \left(Df^{-1}\right)_{f(x)}$$

Then,

$$D\left(f^{-1}\right)_{f(x)} = (Df_x)^{-1}$$

A quick note on notation. $f \colon V \longrightarrow W$ and $f^{-1} \colon W \longrightarrow V$ is misleading because $f \colon A \longrightarrow \mathbb{R}^n$ need not be invertible. But we write $f^{-1}$ for a local inverse. A better notation is $f|_V \colon V \longrightarrow W, (f|_V)^{-1} \colon W \longrightarrow V$ ($f$ restricted to $V$).

**Example 33.** $f \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ ($2 \times 2$ Jacobians), $f(x,y) = (e^x \cos y, e^x \sin y)$. Where is $f$ locally invertible?

$f$ is locally invertible wherever $f^{-1}(x,y)$ is invertible? (By the theorem, wherever the matrix is invertible).

$$f'(x,y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$
$$\det\left(f^{-1}(x,y)\right) = e^{2x}\cos^2 y + e^{2x}\sin^2 y = e^2 x > 0$$

for every $(x,y)$. It follows that $f$ is locally invertible at every $(x,y) \in \mathbb{R}^2$. But $f(0,0) = (1,0) = f(0,2\pi)$ so $f$ is not one-to-one, hence, $f \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is not globally invertible.

### 7.1.1 $2 \times 2$ Invertible Matrices

A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertifle $\iff \det(A) = ad - bc \neq 0$ and

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

From the example, find the best affine approximation to $f^{-1}$ at $f\left(0, \frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$$\begin{aligned} D\left(f^{-1}\right)_{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)} &= \left(Df_{\left(0, \frac{\pi}{4}\right)}\right)^{-1} = \left(f^{-1}\right)'\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left(f'\left(0, \frac{\pi}{4}\right)\right)^{-1} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

So, the best affine approximation is

$$f^{-1}\left(\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) + (h,k)\right) = \begin{bmatrix} 0 \\ \frac{\pi}{4} \end{bmatrix} + \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix}$$

## 7.2 The Implicit Function Theorem

Before introducing the Implicit Function Theorem, we want to understand what does it mean to *implicitly* define something? Consider $x + 2y = 1$, we think of this as a constraint equation on $(x, y)$. We can implicitly define $y$ in terms of $x$ by $y = \frac{1-x}{2}$. The constraint also defines $x$ as a function of $y$.

Alternately, consider the equation of a unit circle, $x^2 + y^2 = 1$. We can't globally define all $y$ as a function of $x$ but we can define it locally as so,

$$\begin{aligned} y &= \pm\sqrt{1 - x^2} \\ x &= \pm\sqrt{1 - y^2} \end{aligned}$$

**Definition 32** (Implicit Function Theorem)**.** Let $0 < c < n$ be integers and set $r = n - c$. Let $A \subseteq \mathbb{R}^n$ be open, and let $g\colon A \longrightarrow \mathbb{R}^c$ be $\mathcal{C}^1$. Let $L = \{v \in A \colon g(v) = \mathbf{0}_c\}$ . Let $p \in L$ and write $p = (a, b) \in \mathbb{R}^r \times \mathbb{R}^c$ and

$$g'(p) = \begin{bmatrix} g'_{(x)}(p) & g'_{(y)}(p) \end{bmatrix} \qquad \left(g'_{(x)}(p) \text{ is } c \times r \text{ and } g'_y(p) \text{ is } c \times c\right)$$

If $g'_y(p)$ is invertible, then $L$ is locally the graph of a function near $p$; more precisely, there is an open $B \subseteq \mathbb{R}^n$ with $p = (a, b) \in B$ and a function $\varphi$ such that

$$g(x, y) = \mathbf{0}_c \iff y = \varphi(x)\,\forall\,(x, y) \in B$$

Moreover, the function $\varphi$ is differentiable at $a$ and

$$\varphi'(a) = -\left(g'_y(p)\right)^{-1}\left(g'_x(p)\right)$$

**Example 34.** $g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, g(x, y) = x^2 + y^2 - 1, g(x, y) = 0 \iff x^2 + y^2 = 1$.

$$g'(x, y) = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

$$g'(0, 1) = \begin{bmatrix} 0 & 2 \end{bmatrix}$$

The submatrix $\begin{bmatrix} 2 \end{bmatrix}$ is invertible and so, $y = \varphi(x) = \sqrt{1 - x^2}$ local to $(0, 1)$.

Putting the $y$ (dependent) variables at the end of the variable list is just a matter of convenience of stating the theorem. We can always change the order of variables without changing properties of the function.

$$g(v) = \underbrace{0_c}_{c \text{ constraint equations, } n = r + c \text{ variables}}$$

Given $p \in L$, we want to find any $c$ variables which can be written as functionsof the other $r$ variables. You can do this for any $c$ variables whose corresponding columns in

$g'(p)$ form an invertible $c \times c$ matrix.

**Example 35.** $x^2 + y^2 + z^2 - 1 = 0 \iff x^2 + y^2 + z^2 = 1$

$$g(x, y, z) = x^2 + y^2 + z^2 - 1, L_0^g = \{(x, y, z) : g(x, y, z) = 0\}$$

$$g'(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

For $p = (0, 01)$,

$$g'(p) = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}$$

We see that near $(0, 0, 1)$ on $L$, $z$ can be written as a function of $x, y$, that is,

$$z = \sqrt{1 - x^2 - y^2}$$

For $p = (0, -1, 0)$,

$$g'(p) = \begin{bmatrix} 0 & -2 & - \end{bmatrix}$$

We see that near $(0, -1, 0)$ on $L$, $y$ can be written as a function of $x, z$, that is

$$y = -\sqrt{1 - x^2 - z^2}$$

For $p = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$,

$$g'(p) = \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{bmatrix}$$

We see that near $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $x, y, z$ can be written as functions of the other two valuesariables since all three submatrices are invertible.

**Example 36.** $g(x, y, z) = \left(x^2 + y^2 + z^2 - 3, y + z - 2\right)$

$g(x, y, z) = (0, 0)$ defined intersection of a sphere and a plane. For $p = (1, 1, 1), g(p) = (0, 0)$.

$$g'(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \\ 0 & 1 & 1 \end{bmatrix}, g'(1, 1, 1) = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

We see that $x, y$ can be written as a function of $z$,

$$\begin{cases} x^2 + y^2 + z^2 = 3 \\ y + z = 2 \end{cases}$$

And so,

$$x = \sqrt{3 - y^2 - z^2} \implies x = \sqrt{3 - (2 - z)^2 - z^2}$$
$$y = 2 - z$$

*Proof. (Implicit Function Theorem $\implies$ Inverse Function Theorem).* We want to invert $f \colon A \longrightarrow \mathbb{R}^n$ (where $A \subseteq \mathbb{R}^n$) (local to some point), that is, solve $f(x) = y$ for $x$ in terms of $y$. Notice that $f(x) = y$ is a constraint equation so apply the implicit function theorem such that $g(x, y) = f(x) - y$. Let $f, g \in A$ be as in statement of the Implicit Function Theorem. Define $g \colon A \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $g(x, y) = f(x) - y$. Note $L = L_0^g = \operatorname{graph}(f)$. Set $b = f(a), g(a, b) = f(a) - b = 0$.

$$\begin{aligned} g'(a, b) &= \begin{bmatrix} g'_{(x)}(a, b) & g'_{(y)}(a, b) \end{bmatrix} \\ &= \begin{bmatrix} \underbrace{f'(a)}_{\text{invertible } n \times n block} & -I_n \end{bmatrix} \end{aligned}$$

By Implicit Function Theorem, there is an open $\mathcal{B}$ with $(a, b) \in \mathcal{B}$ and a function $\varphi$ s.t.

$$f(x) = y \iff \varphi(y) = x \forall (x, y) \in \mathcal{B}$$

In other words, $\varphi = f^{-1}$ on some suitable domain. □

*(Proof. Inverse Function Theorem $\implies$ Implicit Function Theorem).* We want to solve $g(x,y) = 0$ for $y$ variable in terms of $x$ variables. Consider $f(x,y) = (x, g(x,y))$. Suppose we can invert $f$ locally, then $g$ on any $x$

$$(x,y) \longmapsto (x,y) \text{ s.t. } g(x,y) = 0$$

$$f'(x,y) = \begin{bmatrix} I_r & 0 \\ g'_{(x)}(x,y) & g'_{(y)}(x,y) \end{bmatrix}$$

By Inverse Function Theorem, we can locally invert $f$ since the matrix is invertible. □

## 7.3 Lagrange Multipliers

To satisfy constraints is to meet a condition

$$g(x) = 0_c \begin{cases} g_1(x) = 0 \\ \vdots \\ g_c(x) = 0 \end{cases}$$

$$L = L_0^g = L_0^{g_1} \cap L_0^{g_2} \cap \cdots \cap L_0^{g_c}$$

At $p \in L$, $\nabla g_i(p)$ is normal to $L$. We also note that the linear combinations $\sum_{i=1}^{c} b_i \nabla g_i(p)$ are also normal to $L$. Do such vectors describe all vectors normal to $L$? It depends. If the Jacobian matrix $\nabla g(p)$ has *full rank*. We will cover this later.

When optimizing $f$ on $L$, we look at critical points $p$ for $f|_L$ ($f$ restricted to $L$).

If $d$ is a direction on $L$, then, $D_d f(p) = 0$. Assuming differentiability at $p$,

$$\begin{aligned} 0 &= D_d f(p) \\ 0 &= \langle d, \nabla f(p) \rangle \forall d \text{ tangent to } L \\ \implies & \nabla f(p) \perp d \\ \implies & \nabla f(p) \text{ is normal to } L \end{aligned}$$

Thus, $\nabla f(p)$ is a linear combination of $\nabla g_i f(p)$ assuming they represent all normal vectors to $L$. This gives us the Lagrange conditions for optimizer $p$,

$$\begin{cases} \nabla f(p) = \lambda_1 \nabla g_1(p) + \cdots + \lambda_c \nabla g_c(p) \\ g(p) = 0 \end{cases}$$

These provide a necessary condition for optimizer points assuming nice enough properties of $f, g$ at $p$. We can now solve optimization problems.

**Example 37.** Find the point on the intersection of $x+y+z = 1$ and $2x - y + z = -1$, which is closest to the origin?

We rewrite $\min f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $g(x,y,z) = (g_1, g_2) = (x+y+z-1, 2x-y+z+1) = (0,0)$. Then, the Lagrange conditions are

$$\begin{cases} \nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z) \\ x+y+z = 1 \\ 2x - y + z = -1 \end{cases}$$

Thus,

$$\nabla f(x,y,z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}, \nabla g_1(x,y,z) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \nabla g_2(x,y,z) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The 2 gets absorbed into constants,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

### 7.3.1 More on Lagrange Multipliers

**Definition 33** (Full Rank)**.** Let $A$ be a $c \times n, c \leq n$. $A$ has *full rank* if there are columns which form an invertible matrix.

**Example 38.** The matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has columns that form an invertible matrix, so, it has full rank.

**Example 39.** The matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ doesn't have columns that form an invertible matrix, so, it's not full rank.

**Theorem 33** (Lagrange Multiplier Criterion)**.** *Let $0 < c < n$ be integers, $A \subseteq \mathbb{R}^n, g\colon A \longrightarrow \mathbb{R}^c$ be $\mathbb{C}^1$ on all interior points of $A$.*

*Let $L = L_0^g = \{x \in A\colon g(x) = c\}$. Let $f\colon A \longrightarrow \mathbb{R}$. Suppose the restriction of $f$ to $L$ has an extreme value at $p \in L$, where $p$ in interior of $A$. Suppose $f$ is differentiable at $p$ and $g'(p)$ has full rank. Then, the following conditions hold:*

$$\begin{cases} \nabla f(p) = \sum_{i=1}^{c} \lambda_i g_i(p) \\ g(p) = 0 \end{cases}$$

**Definition 34** (Regular Value)**.** 0 is a *regular value* of $f$ if $L_0^g$ contains only interior points of $A$ and $g'(p)$ has full rank for all $p \in L_0^g$.

**Corollary.** In the notation of the theorem, if $f$ is differentiable at all points $p \in L$ and if 0 is a regular value of $g$, then any $p \in L$ which has an extreme value for $f$ restricted to $L$ must satisfy the Lagrange conditions.

**Remark.** So, such points are the only candidates to solve min/max for $f(x)$ subject to $g(x) = 0$.

**Example 40.** Find point on unit circle $x^2 + y^2 = 1$ closest to the line $x + y = 2$.

We change to nomize $\min f(x,y,v,w) = (x-v)^2 + (y-w)^2$ subject to $x^2 + y^2 - 1 = 0, v + w - 2 = 0$.

$$g'(x,y,v,w) = \begin{bmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Note, as long as $(x,y) \neq (0,0)$, $g'(x,y,v,w)$ has full rank $\implies$ 0 is a regular value of $g$. ($(0,0)$ not on unit circle.)

By the corollary, we didn't miss any optimizers. Only looked at points satisgying Lagrange conditions.

**Example 41.** Minimize $f(x,y) = y$, subject to $g(x,y) = y^3 - x^4 = 0$.

Note that $(x,y) \in L = L_0^g \implies y^3 = x^4 > 0 \implies y \geq 0 \implies f(x,y) \geq 0$ on $L$. Also $(0,0) \in L$ so $f(0,0) = 0$ solves the miimization problem.

Trivial.

However, what's important to note here is that the Lagrange conditions, in fact, miss this solution. See below,

$$\begin{cases} \nabla g = \lambda \nabla g \\ y^3 - x^4 = 0 \end{cases} \implies \begin{cases} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -4x^3 \\ 3y^2 \end{bmatrix} \\ y^3 - x^4 = 0 \end{cases}$$

We note that $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -4x^2 \\ 3y^2 \end{bmatrix}$ implies $x = 0, y \neq 0, \lambda \neq 0$ however, $y^3 - x^4 = 0$ implies that $x = 0 \implies y = 0$. There is a contradiction in the value of $y$.

The Lagrange Conditions don't work since $(0,0) \in L$ but $g'(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$ which is not full rank so 0 is not a regular value of $g$ and so, the Lagrange conditions miss the solution.

**Example 42.** $\max f(x_1, \cdots, x_n) = (x_1 \cdots x_n)^{1/n}$ subject to $\dfrac{x_1 + \cdots + x_n}{n} = 1$ and $x_1, \cdots, x_n > 0$.

$g(x_1, \cdots, x_n) = \dfrac{x_1 + \cdots + x_n}{n} - 1$ so,

$$g'(x_1, \cdots, x_n) = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

Note that 0 is a regular value so we only need to look at Lagrange conditions.

$$\begin{aligned} \nabla f(x_1, \cdots, x_n) &= \left( \frac{1}{n} (x_1 \cdots x_n)^{1/n-1} x_2 \cdots x_n, \cdots, \frac{1}{n} (x_1 \cdots x_n)^{1/n} x_1 \cdots x_{n-1} \right) \\ &= \left( \frac{f(x_1, \cdots, x_n)}{nx_1}, \cdots, \frac{f(x_1, \cdots, x_n)}{nx_n} \right) = \frac{f(x_1, \cdots, x_n)}{n} \left( \frac{1}{x_1}, \cdots, \frac{1}{x_n} \right) \end{aligned}$$

$$\begin{aligned} &\nabla f = \lambda \nabla g \\ &\implies \frac{f(x_1, \cdots, x_n)}{nx_i} = \frac{1}{n} \text{ for } i \in [1,n] \\ &\implies f(x_1, \cdots, x_n) = x_i \text{ for } i \in [1,n] \implies x_1 = x_2 = \cdots = x_n \end{aligned}$$

$$\begin{cases} x_1 = \cdots = x_n \\ \dfrac{x_1 + \cdots + x_n}{n} = 1 \end{cases}$$

All $x_i > 0 \implies x_1 = x_2 = \cdots x_n = 1$, then $f$ has max value of 1 at $(1, \cdots, 1)$.

**Corollary.** (Arithmetic-Geometric Mean Inequality) For all positive $a_1, \cdots, a_n$, $(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}$.

*Proof.* Let $a_1, \cdots, a_n > 0$. Set $a = \frac{a_1 + \cdots + a_n}{n}$. Set $x_i = \frac{a_i}{a}$ for $i = 1, \cdots, n$. Then,

$$\frac{x_1 + \cdots + x_n}{n} = \frac{\frac{a_1}{a} + \cdots + \frac{a_n}{a}}{n} = \frac{1}{a} \left( \frac{a_1 + \cdots + a_n}{n} \right) = 1$$

By previous example,

$$(x_1 \cdots x_n)^{1/n} \leq 1$$

Then,

$$\begin{aligned}(a_1 \cdots a_n)^{1/n} &= ((ax_1) \cdots (ax_n))^{1/n} = a\,(x_1 \cdots x_n)^{1/n} \\ &\leq a \cdot 1 = \frac{a_1 + \cdots + a_n}{n}\end{aligned}$$

□

## 7.4 On Determinants and Cross Product

$$\det \begin{bmatrix} a \end{bmatrix} = a$$

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

We can define $n \times n$ determinant recursively. Let the matrix $A$,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\det A = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A+^{(i1)}$$

Essentially cofactor expansion along 1st column. We can do cofactor expansion aong any row or column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

**Theorem 34.** *For any square matrix $A$, $A$ is invertible* $\iff \det A \neq 0$.

**Definition 35** (Cross Product)**.** The cross product is a special operation on $\mathbb{R}^3$ and is defined as

$$\times : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \det \begin{bmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_3 \\ a_3 & b_3 & e_3 \end{bmatrix}$$

Note that the abobe matrix is not a real matrix because some entries are vectors. It follows that,

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

## 7.5 Properties of Cross Product

(1) $\times$ is skew-symmetric: $(u \times v) = -(v \times u)$ (anticommutative)

(2) Bilinearity: $(au + a'u') \times v = a\,(u \times v) + a'\,(u' \times v)$ and $u \times (bv + b'v') = b\,(u \times v) + b'\,(u \times v')$

(3) $u \times v$ is orthogonal to both $u$ and $v$.

(4) $\times$ is 0 $\iff$ $u, v$ are parallel.

(5) $|u \times v|$ = area of the parallelogram spanned by $u$ and $v$.

(6) If $u, v$ not parallel, $[u, v, u \times v]$ is right-hand oriented. $(u \times v)$ is normal to the plane spanned by $u$ and $v$.